

RIGIDITY AROUND POISSON SUBMANIFOLDS

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ABSTRACT. We prove a rigidity theorem in Poisson geometry around compact Poisson submanifolds, using the Nash-Moser fast convergence method. In the case of one-point submanifolds (fixed points), this immediately implies a stronger version of Conn's linearization theorem [1], also proving that Conn's theorem is, indeed, just a manifestation of a rigidity phenomenon; similarly, in the case of arbitrary symplectic leaves, it gives a stronger version of the local normal form theorem [7]; another interesting case corresponds to spheres inside duals of compact semisimple Lie algebras, our result can be used to fully compute the resulting Poisson moduli space [13].

INTRODUCTION

Recall that a **Poisson structure** on a manifold M is a Lie bracket $\{\cdot, \cdot\}$ on the space $C^\infty(M)$ of smooth functions on M which acts as a derivation in each entry:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \quad f, g, h \in C^\infty(M).$$

A Poisson structure can be given also by a bivector $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi, \pi] = 0$ for the Schouten bracket. The Lie bracket is related to π by the formula

$$\langle \pi, df \wedge dg \rangle = \{f, g\}, \quad f, g \in C^\infty(M).$$

The **Hamiltonian** vector field of a function $f \in C^\infty(M)$ is

$$X_f = \{f, \cdot\} \in \mathfrak{X}(M).$$

These vector fields span an involutive singular distribution on M , which integrates to a partition of M into regularly immersed submanifolds called **symplectic leaves**. These leaves are symplectic manifolds, the symplectic structure on the leaf S is given by $\omega_S := \pi|_S^{-1} \in \Omega^2(S)$.

The 0-dimensional symplectic leaves are the points $x \in M$ where π vanishes. At such a fixed point x , the cotangent space $\mathfrak{g}_x = T_x^*M$ carries a Lie algebra structure, called the **isotropy Lie algebra** at x , with bracket given by

$$[d_x f, d_x g] := d_x \{f, g\}, \quad f, g \in C^\infty(M).$$

Conversely, starting from a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ there is an associated Poisson structure $\pi_{\mathfrak{g}}$ on the vector space \mathfrak{g}^* , called **the linear Poisson structure**, defined by

$$\{f, g\}_\xi := \langle \xi, [d_\xi f, d_\xi g] \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*).$$

So, at a fixed point x , the tangent space $T_x M = \mathfrak{g}_x^*$ carries a canonical Poisson structure $\pi_{\mathfrak{g}_x}$ which plays the role of the first order approximation of (M, π) around x in the realm of Poisson geometry. We recall Conn's linearization theorem [1]:

Conn's Theorem. *Let (M, π) be a Poisson manifold and $x \in M$ be a fixed point of π . If the isotropy Lie algebra \mathfrak{g}_x is semisimple of compact type then a neighborhood of x in (M, π) is Poisson-diffeomorphic to a neighborhood of the origin in $(\mathfrak{g}_x^*, \pi_{\mathfrak{g}_x})$.*

Conn's proof is analytic, it uses the fast convergence method of Nash and Moser. A new proof of Conn's theorem, which uses Poisson-geometric techniques, is now available in [6]. This geometric proof was adapted to the case of general symplectic

leaves [7], and the outcome will be explain in the sequel. First recall that the cotangent bundle of a Poisson manifold (M, π) carries a Lie algebroid structure: the anchor is $\pi^\sharp : T^*M \rightarrow TM$ and the Lie bracket is

$$[\alpha, \beta]_\pi = L_{\pi^\sharp(\alpha)}(\beta) - L_{\pi^\sharp(\beta)}(\alpha) - d\pi(\alpha, \beta), \quad \alpha, \beta \in \Gamma(T^*M).$$

The restriction this Lie algebroid to a symplectic leaf (S, ω_S) is a transitive Lie algebroid $A_S := T^*M|_S$ over S .

Conversely, given a transitive Lie algebroid $(A, [\cdot, \cdot], \rho)$ over a symplectic manifold (S, ω_S) , one forms the isotropy bundle of A , $\mathfrak{g}(A) := \ker(\rho)$; similar to the linear Poisson structures, $\mathfrak{g}(A)^*$ carries a Poisson structure π_A , which is well defined on a neighborhood $N(A)$ of S (identified with the zero section), such that (S, ω_S) is a symplectic leaf of $(N(A), \pi_A)$ and A can be recovered as the Lie algebroid corresponding to this leaf: $A \cong A_S$. The Poisson manifold $(N(A), \pi_A)$ was first constructed by Vorobjev [20] to serve as the first order approximation of a Poisson structure around a symplectic leaf.

We recall the following normal form result in this setting (Theorem 1 [7]):

Theorem (The normal form theorem from [7]). *Let (M, π) be a Poisson manifold and (S, ω_S) a compact symplectic leaf. If the Lie algebroid A_S is integrable and the 1-connected Lie groupoid integrating it is compact and its s -fibers have vanishing de Rham cohomology in degree two, then a neighborhood of S in (M, π) is Poisson-diffeomorphic to a neighborhood of the zero section in the local model $(N(A_S), \pi_{A_S})$.*

In the case of fixed points this is equivalent to Conn's result.

The original goal of this research was to apply Conn's analytic techniques to Poisson structures around general symplectic leaves and reprove this theorem. The reason for doing this is that Conn's analytic proof seems stronger than the geometric one; in particular, as suggested by Crainic, it should imply rigidity of the Poisson structure around the fixed point. The precise rigidity property that will be used is:

Definition. *Let (M, π) be a Poisson manifold and $S \subset M$ a compact submanifold. We say that π is C^p - C^1 -**rigid around** S , if there are small enough open neighborhoods U of S , such that for all opens O with $S \subset O \subset \overline{O} \subset U$, there exist*

- *an open neighborhood $\mathcal{V}_O \subset \mathfrak{X}^2(U)$ of $\pi|_U$ in the compact-open C^p -topology,*
- *a function $\tilde{\pi} \mapsto \psi_{\tilde{\pi}}$, which associates to every Poisson structure $\tilde{\pi} \in \mathcal{V}_O$ an open embedding $\psi_{\tilde{\pi}} : \overline{O} \hookrightarrow M$,*

such that $\psi_{\tilde{\pi}}$ is a Poisson diffeomorphism between

$$\psi_{\tilde{\pi}} : (O, \tilde{\pi}|_O) \xrightarrow{\sim} (\psi_{\tilde{\pi}}(O), \pi|_{\psi_{\tilde{\pi}}(O)}),$$

and ψ is continuous at $\tilde{\pi} = \pi$ (with $\psi_\pi = \text{Id}_{\overline{O}}$), with respect to the C^p -topology on the space of Poisson structures and the C^1 -topology on $C^\infty(\overline{O}, M)$.

We prove the following improvement of [7], which also includes rigidity:

Theorem 1. *Let (M, π) be a Poisson manifold and (S, ω_S) a compact symplectic leaf. If the Lie algebroid $A_S := T^*M|_S$ is integrable by a compact Lie groupoid whose s -fibers have vanishing de Rham cohomology in degree two, then*

- (a) *in a neighborhood of S , π is Poisson diffeomorphic to its local model around S ,*
- (b) *π is C^p - C^1 -rigid around S .*

Already in the case of fixed points, the first part of this theorem gives a slight generalization of Conn's result, which cannot be obtained by an immediate adaptation of the arguments in [6, 7]. Namely, a Lie algebra is integrable by a compact group with vanishing second de Rham cohomology if and only if it is compact and its center is at most one-dimensional (see Lemma 2.1). The case when the center

is trivial is Conn's result, and the one-dimensional case is a consequence of a result of Monnier and Zung on smooth Levi decomposition of Poisson manifolds [17].

However, the main advantage of the approach of this paper over [7] is that it allows for a rigidity theorem around an arbitrary Poisson submanifold. Recall that a **Poisson submanifold** is a submanifold S of (M, π) such that π is tangent to S . Of course, the symplectic leaves are the simplest type of Poisson submanifolds. The main result of this paper is the following rigidity theorem for integrable Poisson manifolds.

Theorem 2. *Let (M, π) be a Poisson manifold for which the Lie algebroid T^*M is integrable by a Hausdorff Lie groupoid whose s -fibers are compact and their de Rham cohomology vanishes in degree two. For every compact Poisson submanifold S of M we have that*

- (a) π is C^p - C^1 -rigid around S ,
- (b) up to isomorphism, π is determined around S by its first order jet at S .

We prove Theorem 1 by applying part (b) of this result to the local model.

In both theorems, p has the following (most probably not optimal) value:

$$p = 7(\lfloor \dim(M)/2 \rfloor + 5).$$

In part (b) of Theorem 2 we prove that every Poisson structure $\tilde{\pi}$, defined on an open containing S , which satisfies $j_{|S}^1 \pi = j_{|S}^1 \tilde{\pi}$, is isomorphic to π around S by a diffeomorphism which is the identity on S up to first order.

The structure encoded by the first order jet of π at S can be organized as an extension of Lie algebroids (see Remark 2.2 [12])

$$(1) \quad 0 \longrightarrow \nu_S^* \longrightarrow T^*M|_S \longrightarrow T^*S \longrightarrow 0,$$

where $\nu_S^* \subset T^*M|_S$ is the conormal bundle and T^*S is the cotangent Lie algebroid of the Poisson manifold $(S, \pi|_S)$. With this, Theorem 1 follows easily from Theorem 2: if S is a compact symplectic leaf, then the Poisson structures (M, π) and $(N(A_S), \pi_{A_S})$ have the same first order jet around S (they induce the same exact sequence (1)); moreover, the hypothesis of Theorem 1 implies that Theorem 2 can be applied to the local model $(N(A_S), \pi_{A_S})$ (see Lemma 1.2).

One might try to follow the same line of reasoning and use Theorem 2 to prove a normal form theorem around Poisson submanifolds. Unfortunately, around general Poisson submanifolds, a first order local model does not seem to exist. Actually, there are Lie algebroid extensions as in (1) which do not arise as the first jet of Poisson structures (see Example 2.3 in [12]). Nevertheless, one can use Theorem 2 to prove normal form results around particular classes of Poisson submanifolds.

The paper is organized as follows. In section 1, after recalling some properties of Lie groupoids and Lie algebroids, we describe in detail the local model around a leaf and a symplectic groupoid integrating it. We end the section by proving that Theorem 2 implies Theorem 1. Section 2 is an extended introduction to the paper, we give a list of applications, examples and connections with related literature. In section 3 we apply the Nash-Moser method and proof of Theorem 2. The appendix contains three general results on Lie groupoids: existence of invariant tubular neighborhoods, that ideals are integrable (as representations) and the Tame Vanishing Lemma. This last result provides tame homotopy operators for Lie algebroid cohomology with coefficients and, when combined with the Nash-Moser techniques, it is a very useful tool for handling similar geometric problems (see [14]).

About the proof. The proof of the rigidity theorem is inspired mainly by Conn's paper [1]. Conn uses a technique due to Nash and Moser to construct a sequence

of changes of coordinates in which π converges to the linear Poisson structure $\pi_{\mathfrak{g}_x}$. At every step the new coordinates are found by solving some equations which are regarded as belonging to the complex computing the Poisson cohomology of $\pi_{\mathfrak{g}_x}$. To account for the “loss of derivatives” phenomenon during this procedure he uses smoothing operators. Finally, he proves uniform convergence of these changes of coordinates and of their higher derivatives on some ball around x .

Conn’s proof has been formalized in [15, 17] into an abstract Nash Moser normal form theorem. It is likely that part (a) of our Theorem 2 could be proven using Theorem 6.8 in [15]. Due to some technical issues (see Remark 2), we cannot apply this result to conclude neither part (b) of our Theorem 2 nor the normal form Theorem 1, therefore we follow a direct approach.

We also simplified Conn’s argument by giving coordinate free statements and working with flows of vector fields. For the expert: we gave up on the polynomial-type inequalities using instead only inequalities which assert tameness of certain maps, i.e. we work in Hamilton’s category of tame Fréchet spaces. Our proof deviates the most from Conn’s when constructing the homotopy operators. Conn recognizes the Poisson cohomology of $\pi_{\mathfrak{g}_x}$ as the Chevalley-Eilenberg cohomology of \mathfrak{g}_x with coefficients in the Fréchet space of smooth functions. By passing to the Lie group action on the corresponding Sobolev spaces, he proves existence of tame (in the sense of Hamilton [10]) homotopy operators for this complex. We, on the other hand, regard this cohomology as Lie algebroid cohomology, and prove a general tame vanishing result for the cohomology of Lie algebroids integrable by groupoids with compact s -fibers. This is done by further identifying this complex with the invariant part of the de Rham complex of s -foliated forms on the Lie groupoid, and then we use the fiberwise inverse of the Laplace-Beltrami operator to construct the homotopy operators.

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1. PROOF OF THE NORMAL FORM THEOREM (THEOREM 2 \Rightarrow THEOREM 1)

In this section we prove Theorem 1 using Theorem 2. For this, we show that the normal model constructed out of an integrable Lie algebroid is integrable.

1.1. Lie groupoids and Lie algebroids. We recall here some standard results about Lie groupoids and Lie algebroids, for definitions and other basic properties we recommend [11, 16]. To fix notations, the anchor of a Lie algebroid $A \rightarrow M$ will be denoted by ρ ; the source and target maps of a Lie groupoid $\mathcal{G} \rightrightarrows M$ by s and t respectively, the unit map by u .

A Lie groupoid $\mathcal{G} \rightrightarrows M$ has an associated Lie algebroid $A(\mathcal{G})$ over M ; as a vector bundle $A(\mathcal{G})$ is the restriction to M (i.e. pullback by u) of the subbundle $T^s\mathcal{G}$ of $T\mathcal{G}$ consisting of vectors tangent to the s -fibers. The anchor is given by the differential of t . The Lie bracket comes from the identification between sections of $A(\mathcal{G})$ and right invariant vector fields on \mathcal{G} .

A Lie algebroid $(A, [\cdot, \cdot], \rho)$ is called **integrable** if it is isomorphic to the Lie algebroid $A(\mathcal{G})$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$. Not every Lie algebroid is integrable (see [2]). If a Lie algebroid is integrable, then, as for Lie algebras, there exists up to isomorphism a unique Lie groupoid with 1-connected s -fibers integrating it.

A Lie algebroid $A \rightarrow M$ is called **transitive** if ρ is surjective. A Lie groupoid is called **transitive** if the map $(s, t) : \mathcal{G} \rightarrow M \times M$ is a surjective submersion. If \mathcal{G} is

transitive then also $A(\mathcal{G})$ is transitive. Conversely, if $A \rightarrow M$ is transitive and M is connected, then every Lie groupoid integrating it is transitive as well.

Out of a principal bundle $q : P \rightarrow S$ with structure group G one can construct a transitive Lie groupoid $\mathcal{G}(P)$, called the **gauge groupoid** of P , as follows:

$$\mathcal{G}(P) := P \times_G P \rightrightarrows S,$$

with structure maps given by

$$s([p_1, p_2]) := q(p_2), \quad t([p_1, p_2]) := q(p_1), \quad [p_1, p_2][p_2, p_3] := [p_1, p_3].$$

The Lie algebroid of $\mathcal{G}(P)$ is TP/G , where the Lie bracket is obtained by identifying sections of TP/G with G -invariant vector fields on P . Conversely, every transitive Lie groupoid \mathcal{G} is the gauge groupoid of a principal bundle: the bundle is any s -fiber of \mathcal{G} and the structure group is the isotropy group. So, a transitive Lie algebroid A is integrable if and only if there exists a principal G -bundle P such that A is isomorphic to TP/G .

A **symplectic groupoid** is a Lie groupoid endowed with a symplectic structure $(\mathcal{G}, \omega) \rightrightarrows M$, for which the graph of the multiplication is a Lagrangian submanifold:

$$\{(g_1, g_2, g_3) : g_1 g_2 = g_3\} \subset (\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}, pr_1^*(\omega) + pr_2^*(\omega) - pr_3^*(\omega)).$$

This condition is very strong, it implies that the base carries a Poisson structure π such that source map is Poisson and the target map is anti-Poisson. Moreover, \mathcal{G} integrates the cotangent Lie algebroid T^*M of π . Conversely, if for a Poisson manifold (M, π) the Lie algebroid T^*M is integrable, then the s -fiber 1-connected Lie groupoid integrating it is symplectic (see [3]).

1.2. The local model. Consider a Poisson manifold (M, π) and let (S, ω_S) be an embedded symplectic leaf. The structure relevant for our entire discussion is

(2) the symplectic manifold (S, ω_S) and the Lie algebroid $A_S := T^*M|_S$.

The local model of π around S , as constructed by Vorobjev [20], is a Poisson manifold $(N(A_S), \pi_{A_S})$, where $N(A_S)$ is an open of S (viewed as of the zero section) in the normal bundle $\nu_S := TM|_S/TS$, with the following properties (see [14]):

- π_{A_S} is constructed using only the data (2),
- there exists a tubular neighborhood $\Psi : \nu_S \rightarrow M$ such that $\Psi_*(\pi_{A_S})$, the push forward of π_{A_S} , has the same first order jet as π along S .

On the other hand, the first order jet of π along S encodes precisely the data (2) (see Proposition 1.10 in [7]).

When A_S is integrable, we describe the local model more explicitly. In this case, because A_S is transitive, it is isomorphic to TP/G for a principal G -bundle $P \rightarrow S$. So, the relevant data (2) becomes a principal G -bundle $p : P \rightarrow S$ over a symplectic manifold (S, ω_S) . To describe the local model, let $\theta \in \Omega^1(P, \mathfrak{g})$ be a principal connection on P , where \mathfrak{g} denotes the Lie algebra of G . Consider the closed G -invariant 2-form ω on $P \times \mathfrak{g}^*$ given by

$$\omega = p^*(\omega_S) - d\tilde{\theta}, \quad \text{where } \tilde{\theta}_{q,\xi} := \langle \xi, \theta_q \rangle.$$

The open set Σ where ω is nondegenerate is G -invariant and contains $P \times \{0\}$. We have that the map $\mu : \Sigma \rightarrow \mathfrak{g}^*$, $\mu(p, \xi) = \xi$ is an equivariant moment map for the action of G . The local model is obtained by symplectic reduction:

$$(N(P), \pi_P) := (\Sigma, \omega)/G,$$

where $N(P) := \Sigma/G$ is an open neighborhood of the zero section in the associated coadjoint bundle $P[\mathfrak{g}^*]$. Moreover, $N(P)$ has (S, ω_S) as a symplectic leaf (regarded as $P \times \{0\}/G$), and the Lie algebroid $T^*N(P)|_S$ is isomorphic to TP/G . For the proofs of all these statements see [18, 7, 14].

The resulting Poisson manifold $(N(P), \pi_P)$ is integrable, and we describe a symplectic groupoid integrating it. Since this result fits in a more general framework, we state the following lemma, inspired by [8]:

Lemma 1.1. *Let (Σ, ω) be a symplectic manifold endowed with a proper, free Hamiltonian action of a Lie group G and equivariant moment map $\mu : \Sigma \rightarrow \mathfrak{g}^*$. Then Σ/G is integrable and a symplectic Lie groupoid integrating it is*

$$\mathcal{G}(\Sigma) := (\Sigma \times_\mu \Sigma)/G \rightrightarrows \Sigma/G,$$

with symplectic structure Ω induced by $(s^*(\omega) - t^*(\omega))|_{\Sigma \times_\mu \Sigma}$.

Proof. Consider the symplectic groupoid $\mathcal{G} := \Sigma \times \Sigma \rightrightarrows \Sigma$, with symplectic structure $\tilde{\Omega} := s^*(\omega) - t^*(\omega)$. Then G acts on \mathcal{G} by symplectic groupoid automorphism with equivariant moment map $J := s^*\mu - t^*\mu$, which is also a groupoid cocycle. By Proposition 4.6 in [8], the Marsden-Weinstein reduction

$$\mathcal{G}/G = J^{-1}(0)/G$$

is a symplectic groupoid integrating the Poisson manifold Σ/G . In our case $J^{-1}(0) = \Sigma \times_\mu \Sigma$ and the symplectic form Ω is obtained as follows: the 2-form $\tilde{\Omega}|_{\Sigma \times_\mu \Sigma}$ is G -invariant and its kernel is spanned by the vectors coming from the infinitesimal action of \mathfrak{g} . Therefore it is the pullback of a symplectic form Ω on $(\Sigma \times_\mu \Sigma)/G$. \square

In our setting, the lemma shows that the groupoid integrating the local model $(N(P), \pi_P)$ is just the restriction to $N(P)$ of the action groupoid

$$\mathcal{G} := (P \times P \times \mathfrak{g}^*)/G \rightrightarrows P[\mathfrak{g}^*],$$

corresponding to the representation of $P \times_G P$ on $P[\mathfrak{g}^*]$. Observe that if P is compact, then $N(P)$ contains arbitrarily small opens of the form $P[V] := (P \times V)/G$, where V is a G -invariant neighborhood of 0 in \mathfrak{g}^* . These opens are \mathcal{G} -invariant, and the restriction of \mathcal{G} to $P[V]$ is $(P \times P \times V)/G$. In particular, all its s -fibers are diffeomorphic to P . This proves the following:

Proposition 1.2. *The local model $(N(P), \pi_P)$ associated to a principal bundle P over a symplectic manifold (S, ω_S) is integrable by a Hausdorff symplectic Lie groupoid. If P is compact, then there are arbitrarily small invariant neighborhoods U of S , such that all s -fibers over points in U are diffeomorphic to P .*

1.3. Proof of Theorem 2 \Rightarrow Theorem 1. Since $A_S := T^*M|_S$ is transitive, the Lie groupoid integrating it is isomorphic to the gauge groupoid $P \times_G P$ of a principal G -bundle $P \rightarrow S$. Using the isomorphism of Lie algebroids $A_S \cong TP/G$, we obtain an identification of the short exact sequences:

$$0 \longrightarrow P[\mathfrak{g}] \longrightarrow TP/G \longrightarrow TS \longrightarrow 0$$

$$0 \longrightarrow \nu_S^* \longrightarrow A_S \longrightarrow TS \longrightarrow 0.$$

This gives also an isomorphism between $\nu_S \cong P[\mathfrak{g}^*]$. A splitting of the first sequence is a principal connection on P , and a splitting of the second is equivalent to an inclusion of ν_S in $TM|_S$. Consider a tubular neighborhood of S in M

$$\Psi : \nu_S \longrightarrow M,$$

and denote by θ the principal connection induced by its differential along S . The normal model $(N(P), \pi_P)$ constructed with the aid of θ has the property that π and $\Psi_*(\pi_P)$ have the same first order jet along S (see [14] for details). By Proposition 1.2 there exists $U \subset N(P)$, an open neighborhood of S , for which $\pi_{P|U}$ is integrable by a Hausdorff Lie groupoid whose s -fibers are diffeomorphic to P . Since P is compact and $H^2(P) = 0$, the Poisson manifold $(U, \pi_{P|U})$ satisfies the conditions of Theorem 2. By part (a), $\pi_{P|U}$ is C^p - C^1 -rigid around S , and by part (b) π_P and π are Poisson diffeomorphic around S , thus π is also C^p - C^1 -rigid around S .

2. REMARKS, EXAMPLES AND APPLICATIONS

In this section we give a list of examples and applications for our two theorems and we also show some links with other results from the literature.

2.1. A global conflict. Theorem 2 does not exclude the case when the Poisson submanifold S is the total space M , and the conclusion is that a compact Poisson manifold (M, π) for which the Lie algebroid T^*M is integrable by a compact groupoid \mathcal{G} whose s -fibers have vanishing H^2 is globally rigid. Nevertheless, this result is useless, since the only example of such a manifold is S^1 , for which the trivial Poisson structure is clearly rigid. In the case when \mathcal{G} has 1-connected s -fibers, this conflict was pointed out in [4], for the general case, see [14].

2.2. C^p - C^1 -rigidity and isotopies. In the definition of C^p - C^1 -rigid, we may assume that the maps $\psi_{\bar{\pi}}$ are isotopic to the inclusion $\text{Id}_{\bar{\mathcal{O}}}$ of $\bar{\mathcal{O}}$ in M by a path of diffeomorphisms. This follows from the continuity of ψ and the fact that $\text{Id}_{\bar{\mathcal{O}}}$ has a path connected C^1 -neighborhood in $C^\infty(\bar{\mathcal{O}}, M)$ consisting of open embeddings (for details, see [14]).

2.3. A comparison with the local normal form theorem from [7]. Part (a) of Theorem 1 is a slight improvement of the normal form result from [7]. Both theorems require the same conditions on a Lie groupoid, for us this groupoid could be any integration of A_S , but in *loc.cit.* it has to be the Weinstein groupoid of A_S (i.e. the s -fiber 1-connected). In the sequel we will study two extreme examples which already reveal the wider applicability of Theorem 1: the case of fixed points and the case of Poisson structures with trivial underlying foliation. For more examples, see section 2 in *loc.cit.*

2.4. The case of fixed points. Consider a Poisson manifold (M, π) and let $x \in M$ be a fixed point of π . In a chart centered at x , we write

$$(3) \quad \pi = \sum_{i,j} \frac{1}{2} \pi_{i,j}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \text{ with } \pi_{i,j}(0) = 0.$$

The local model of π around 0 is given by its first jet at 0:

$$\sum_{i,j,k} \frac{1}{2} \frac{\partial \pi_{i,j}}{\partial x_k}(0) x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The coefficients $C_{i,j}^k := \frac{\partial \pi_{i,j}}{\partial x_k}(0)$ are the structure constants of the isotropy Lie algebra \mathfrak{g}_x (see the Introduction). To apply Theorem 1 in this setting, we need that \mathfrak{g}_x is integrable by a compact Lie group with vanishing second de Rham cohomology. Such Lie algebras have the following structure:

Lemma 2.1. *A Lie algebra \mathfrak{g} is integrable by a compact Lie group with vanishing second de Rham cohomology if and only if it is of the form*

$$\mathfrak{g} = \mathfrak{k} \text{ or } \mathfrak{g} = \mathfrak{k} \oplus \mathbb{R},$$

where \mathfrak{k} is a semisimple Lie algebra of compact type.

Proof. It is well known that a compact Lie algebra \mathfrak{g} decomposes as a direct product $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{z}$, where $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple of compact type and \mathfrak{z} is the center of \mathfrak{g} . Let G be a compact, connected Lie group integrating \mathfrak{g} . Its cohomology can be computed using invariant forms, hence $H^1(G) \cong H^1(\mathfrak{g})$. By Hopf's theorem G is homotopy equivalent to a product of odd-dimensional spheres, therefore $H^2(G) = \Lambda^2 H^1(G)$. This shows that:

$$(4) \quad H^2(G) = \Lambda^2 H^1(G) \cong \Lambda^2 H^1(\mathfrak{g}) = \Lambda^2(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = \Lambda^2 \mathfrak{z}^*.$$

So $H^2(G) = 0$ implies that $\dim(\mathfrak{z}) \leq 1$.

Conversely, let K be the compact, 1-connected Lie group integrating \mathfrak{k} . Take $G = K$ in the first case and $G = K \times S^1$ in the second. By (4), $H^2(G) = 0$. \square

So, for fixed points, Theorem 1 gives:

Corollary 2.2. *Let (M, π) be a Poisson manifold with a fixed point x for which the isotropy Lie algebra \mathfrak{g}_x is compact and its center is at most one-dimensional. Then π is rigid around x and an open around x is Poisson diffeomorphic to an open around 0 in the linear Poisson manifold $(\mathfrak{g}_x^*, \pi_{\mathfrak{g}_x})$.*

The linearization result in the semisimple case is Conn's theorem [1] and the case when the isotropy has a one-dimensional center is a consequence of the smooth Levi decomposition theorem of Monnier and Zung [17].

This fits into Weinstein's notion of a nondegenerate Lie algebra [21]. Recall that a Lie algebra \mathfrak{g} is called **nondegenerate**, if every Poisson structure which has isotropy Lie algebra \mathfrak{g} at a fixed point x , is Poisson-diffeomorphic around x to the linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ around 0.

A Lie algebra \mathfrak{g} , for which $\pi_{\mathfrak{g}}$ is rigid around 0, is necessarily nondegenerate. To see this, consider π a Poisson bivector given by (3), whose linearization at 0 is $\pi_{\mathfrak{g}}$. We have a smooth path of Poisson bivectors π^t , with $\pi^1 = \pi$ and $\pi^0 = \pi_{\mathfrak{g}}$, given by

$$\pi^t := t\mu_t^*(\pi) = \sum_{i,j} \frac{1}{2t} \pi_{i,j}(tx) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad t \in [0, 1],$$

where μ_t denotes multiplication by $t > 0$. If $\pi_{\mathfrak{g}}$ is rigid around 0, then, for some $r > 0$ and some $t > 0$, there is a Poisson isomorphism between

$$\psi : (B_r, \pi^t) \xrightarrow{\sim} (\psi(B_r), \pi_{\mathfrak{g}}).$$

Now $\xi := \psi(0)$ is a fixed point of $\pi_{\mathfrak{g}}$, which is the same as an element in $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. It is easy to see that translation by ξ is a Poisson isomorphism of $\pi_{\mathfrak{g}}$, therefore, replacing ψ with $\psi - \xi$, we may assume that $\psi(0) = 0$. By linearity of $\pi_{\mathfrak{g}}$, we have that $\mu_t^*(\pi_{\mathfrak{g}}) = \frac{1}{t}\pi_{\mathfrak{g}}$, and this shows that

$$\pi = \frac{1}{t}\mu_{1/t}^*(\pi^t) = \frac{1}{t}\mu_{1/t}^*(\psi^*(\pi_{\mathfrak{g}})) = \mu_{1/t}^* \circ \psi^* \circ \mu_t^*(\pi_{\mathfrak{g}}),$$

which implies that π is linearizable by the map

$$\mu_t \circ \psi \circ \mu_{1/t} : (B_{tr}, \pi) \longrightarrow (t\psi(B_r), \pi_{\mathfrak{g}}),$$

which maps 0 to 0. Thus \mathfrak{g} is nondegenerate.

2.5. The Poisson sphere in \mathfrak{g}^* . Let \mathfrak{g} be a semisimple Lie algebra of compact type and let G be the compact, 1-connected Lie group integrating it. The linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ is integrable by the symplectic groupoid $(T^*G, \omega_{\text{can}}) \rightrightarrows \mathfrak{g}^*$, with source and target map given by left and right trivialization. All s -fibers of T^*G are diffeomorphic to G and, since $H^2(G) = 0$, we can apply Theorem 2 to any compact Poisson submanifold of \mathfrak{g}^* . An example of such a submanifold is the sphere $\mathbb{S}(\mathfrak{g}^*) \subset \mathfrak{g}^*$ with respect to some invariant metric. We obtain the following result, whose formal version appeared in [12] and served as an inspiration.

Proposition 2.3. *Let \mathfrak{g} be a semisimple Lie algebra of compact type and denote by $\mathbb{S}(\mathfrak{g}^*) \subset \mathfrak{g}^*$ the unit sphere centered at the origin with respect to some invariant inner product. Then $\pi_{\mathfrak{g}}$ is C^p - C^1 -rigid around $\mathbb{S}(\mathfrak{g}^*)$ and, up to isomorphism, it is determined around $\mathbb{S}(\mathfrak{g}^*)$ by its first order jet.*

Using this rigidity result, one can describe an open around $\pi_S := \pi_{\mathfrak{g}|\mathbb{S}(\mathfrak{g}^*)}$ in the moduli space of all Poisson structures on the sphere $\mathbb{S}(\mathfrak{g}^*)$. More precisely, any Poisson structure on $\mathbb{S}(\mathfrak{g}^*)$ which is C^p -close to π_S is Poisson diffeomorphic to one of the type $f\pi_S$, where f is a positive Casimir. If the metric is $\text{Aut}(\mathfrak{g})$ -invariant, then two structures of this type $f_1\pi_S, f_2\pi_S$ are isomorphic if and only if $f_1 = f_2 \circ \chi^*$, for some outer automorphism χ of the Lie algebra \mathfrak{g} . The details are given in [13].

2.6. Relation with stability of symplectic leaves. Recall from [5] that a symplectic leaf (S, ω_S) of a Poisson manifold (M, π) is said to be C^p -**strongly stable** if for every open U around S there exists an open neighborhood $\mathcal{V} \subset \mathfrak{X}^2(U)$ of $\pi|_U$ with respect to the compact-open C^p -topology, such that every Poisson structure in \mathcal{V} has a leaf symplectomorphic to (S, ω_S) . Recall also

Theorem (Theorem 2.2 in [5]). *If S is compact and the Lie algebroid $A_S := T^*M|_S$ satisfies $H^2(A_S) = 0$, then S is a strongly stable leaf.*

If π is C^p - C^1 -rigid around S , then S is a strongly stable leaf. Also, the hypothesis of our Theorem 1 imply those of Theorem 2.2 in *loc.cit.*. To see this, let $P \rightarrow S$ be a principal G -bundle for which $A_S \cong TP/G$. Then

$$H^\bullet(A_S) \cong H^\bullet(\Omega(P)^G).$$

If G is compact and connected, and if $H^2(P) = 0$, then $H^2(A_S) = 0$, because

$$H^\bullet(\Omega(P)^G) \cong H^\bullet(P)^G \subset H^\bullet(P).$$

On the other hand, $H^2(A_S) = 0$ doesn't imply rigidity, counterexamples can be found even for fixed points. Weinstein proves [22] that a noncompact semisimple Lie algebra \mathfrak{g} of real rank at least two is degenerate, so $\pi_{\mathfrak{g}}$ is not rigid (see subsection 2.4). However, 0 is a stable point for $\pi_{\mathfrak{g}}$, because by Whitehead's Lemma $H^2(\mathfrak{g}) = 0$.

According to Theorem 2.3 in [5], the condition $H^2(A_S) = 0$ is also necessary for strong stability of the symplectic leaf (S, ω_S) for Poisson structures of "first order", i.e. for Poisson structures which are isomorphic to their local model around S . So, for this type of Poisson structures, $H^2(A_S) = 0$ is also necessary for rigidity.

For Poisson structures with trivial underlying foliation, we will prove below that the hypotheses of Theorem 1 and of Theorem 2.2 *loc.cit.* are equivalent.

2.7. Trivial symplectic foliations. We will discuss now rigidity and linearization of regular Poisson structures π on $S \times \mathbb{R}^n$ with symplectic foliation

$$\{(S \times \{y\}, \omega_y := \pi_{|S \times \{y\}}^{-1})\}_{y \in \mathbb{R}^n},$$

where ω_y are symplectic forms on S . Let (S, ω_S) be the symplectic leaf for $y = 0$. The local model around S corresponds to the family of 2-forms (see [14])

$$j_S^1(\omega)_y := \omega_S + \delta_S \omega_y,$$

where $\delta_S \omega_y$ is the "vertical derivative" of ω

$$\delta_S \omega_y := \frac{d}{d\epsilon} (\omega_{\epsilon y})|_{\epsilon=0} = y_1 \omega_1 + \dots + y_n \omega_n.$$

The local model is defined on an open $U \subset S \times \mathbb{R}^n$ containing S , such that $j_S^1(\omega)_y$ is nondegenerate along $U \cap (S \times \{y\})$, for all $y \in \mathbb{R}^n$. Using the splitting $T^*M|_S = T^*S \times \mathbb{R}^n$ and the isomorphism of $\omega_S^\sharp : TS \xrightarrow{\sim} T^*S$, we identify $A_S \cong TS \oplus \mathbb{R}^n$. With this, the Lie bracket becomes (see [14])

$$\begin{aligned} (5) \quad & [(X, f_1, \dots, f_n), (Y, g_1, \dots, g_n)] = \\ & = ([X, Y], X(g_1) - Y(f_1) + \omega_1(X, Y), \dots, X(g_n) - Y(f_n) + \omega_n(X, Y)). \end{aligned}$$

The conditions in Theorem 1 become more computable in this case.

Lemma 2.4. *If S is compact, then the following are equivalent:*

- (a) A_S is integrable by a compact principal bundle P , with $H^2(P) = 0$,
- (b) $H^2(A_S) = 0$,
- (c) The cohomological variation $[\delta_S \omega] : \mathbb{R}^n \rightarrow H^2(S)$ satisfies:
 - (c₁) it is surjective,
 - (c₂) its kernel is at most 1-dimensional,
 - (c₃) the map $H^1(S) \otimes \mathbb{R}^n \rightarrow H^3(S)$, $\eta \otimes y \mapsto \eta \wedge [\delta_S \omega_y]$ is injective.

Proof. The fact that (a) implies (b) was explained in subsection 2.6. We show now that (b) and (c) are equivalent. The complex computing $H^\bullet(A_S)$ is given by:

$$\Omega^k(A_S) := \bigoplus_{p+q=k} \Omega^p(S) \otimes \Lambda^q \mathbb{R}^n,$$

and the differential acts on elements of degree one and two as follows:

$$\begin{aligned} d_{A_S}(\lambda, \sum_i \mu_i e_i) &= (d\lambda - \sum_i \mu_i \omega_i, \sum_i d\mu_i \otimes e_i, 0), \\ d_{A_S}(\alpha, \sum_i \beta_i \otimes e_i, \sum_{i,j} \gamma_{i,j} e_i \wedge e_j) &= \\ &= (d\alpha + \sum_i \beta_i \wedge \omega_i, \sum_i (d\beta_i - \sum_j \gamma_{i,j} \omega_j) \otimes e_i, \sum_{i,j} d\gamma_{i,j} \otimes e_i \wedge e_j, 0), \end{aligned}$$

where e_1, \dots, e_n is the canonical basis of \mathbb{R}^n . We will use the exact sequences:

$$\begin{aligned} 0 \longrightarrow K \longrightarrow H^2(A_S) \longrightarrow \Lambda^2 \mathbb{R}^n \longrightarrow H^2(S) \otimes \mathbb{R}^n, \\ \mathbb{R}^n \longrightarrow H^2(S) \longrightarrow K \longrightarrow H^1(S) \otimes \mathbb{R}^n \longrightarrow H^3(S), \end{aligned}$$

where $H^2(A_S) \rightarrow \Lambda^2 \mathbb{R}^n$ sends $[\alpha, \beta \otimes u, v \wedge w] \mapsto v \wedge w$; the map $\Lambda^2 \mathbb{R}^n \rightarrow H^2(S) \otimes \mathbb{R}^n$, which we denote by $[\delta_S \omega] \otimes \text{Id}$, sends $v \wedge w \mapsto [\delta_S \omega_v] \otimes w - [\delta_S \omega_w] \otimes v$; the map $\mathbb{R}^n \rightarrow H^2(S)$ is $[\delta_S \omega]$; the map $H^2(S) \rightarrow K$ sends $[\alpha] \mapsto [\alpha, 0, 0]$; the map $K \rightarrow H^1(S) \otimes \mathbb{R}^n$ is $[\alpha, \beta \otimes v, 0] \mapsto [\beta] \otimes v$; and the last map is the one from (c₃). When proving exactness, the only nontrivial part of the computation is at $\Lambda^2 \mathbb{R}^n$. This is based on a simple fact from linear algebra:

$$(6) \quad \ker([\delta_S \omega] \otimes \text{Id}) = \Lambda^2(\ker[\delta_S \omega]).$$

So, an element in $\ker([\delta_S \omega] \otimes \text{Id})$ can be written as a sum of the form $\sum v \wedge w$, with $v, w \in \ker[\delta_S \omega]$. Writing $\delta_S \omega_v = d\eta$, $\delta_S \omega_w = d\theta$, for $\eta, \theta \in \Omega^1(S)$, one easily checks that

$$(\eta \wedge \theta, \eta \otimes w - \theta \otimes v, v \wedge w) \in \Omega^2(A_S)$$

is closed. This implies exactness at $\Lambda^2 \mathbb{R}^n$. So $H^2(A_S)$ vanishes if and only if (c₁), (c₃) hold and the map $[\delta_S \omega] \otimes \text{Id}$ is injective; by (6), injectivity is equivalent to (c₂).

We prove that (b) and (c) imply (a). Part (c₁) implies that, by taking a different basis of \mathbb{R}^n , we may assume that $[\omega_1], \dots, [\omega_n] \in H^2(S, \mathbb{Z})$. Let $P \rightarrow S$ be a principal T^n bundle with connection $(\theta_1, \dots, \theta_n)$ and curvature $(-\omega_1, \dots, -\omega_n)$. We claim that the Lie algebroid TP/T^n is isomorphic to A_S , and therefore A_S is integrable by the compact gauge groupoid

$$P \times_{T^n} P \rightrightarrows S.$$

A section of TP/T^n (i.e. a T^n -invariant vector field on P) can be decomposed uniquely as

$$\tilde{X} + \sum f_i \partial_{\theta_i},$$

where \tilde{X} is the horizontal lift of a vector field X on S , f_1, \dots, f_n are functions on S and ∂_{θ_i} is the unique vertical vector field on P which satisfies

$$\theta_j(\partial_{\theta_i}) = \delta_{i,j}.$$

Using (5) for the bracket on A_S and that $d\theta_i = -p^*(\omega_i)$, it is straightforward to check that the following map is a Lie algebroid isomorphism

$$TP/T^n \xrightarrow{\sim} A_S, \quad \tilde{X} + \sum f_i \partial_{\theta_i} \mapsto (X, f_1, \dots, f_n).$$

Since T^n acts trivially on $H^2(P)$, by (b), we obtain the conclusion:

$$H^2(P) = H^2(P)^{T^n} \cong H^2(TP/T^n) \cong H^2(A_S) = 0.$$

□

So, for trivial symplectic foliations, the conditions in Theorem 1 and in Theorem 2.2 [5] coincide.

Corollary 2.5. *Let $\{\omega_y \in \Omega^2(S)\}_{y \in \mathbb{R}^n}$ be a smooth family of symplectic structures on a compact manifold S . If the cohomological variation at 0*

$$[\delta_S \omega] : \mathbb{R}^n \longrightarrow H^2(S),$$

satisfies the conditions from Lemma 2.4, then the Poisson manifold

$$(S \times \mathbb{R}^n, \{\omega_y^{-1}\}_{y \in \mathbb{R}^n})$$

is isomorphic to its local model and C^p - C^1 -rigid around $S \times \{0\}$.

In the case of trivial foliations we also have an improvement compared to the result of [7]; the hypothesis in there can be restated as (see [14])

- S is compact with finite fundamental group,
- the map $p^* \circ [\delta_S \omega] : \mathbb{R}^n \rightarrow H^2(\tilde{S})$ is an isomorphism,

where $p : \tilde{S} \rightarrow S$ is the universal cover of S . So, for example when S is simply connected, the difference between the assumptions is that, in our case, the map $[\delta_S \omega]$ might still have a 1-dimensional kernel, whereas in [7] it has to be injective.

3. PROOF OF THEOREM 2

We start by preparing the setting needed for the Nash-Moser method: we fix norms on the spaces that we will work with, we construct smoothing operators adapted to our problem and recall the interpolation inequalities. Next, we prove a series of inequalities which assert tameness of some natural operations like Lie derivative, flow and pullback and we also prove some properties of local diffeomorphisms. We end the section with the proof of Theorem 2, which is mostly inspired by [1].

Remark 1. A usual convention when dealing with the Nash-Moser techniques (e.g. [10]), which we also adopt, is to denote all constants by the same symbol. In the series of preliminary results below we work with “big enough” constants C and C_n , and with “small enough” constants $\theta > 0$; these depend on the trivialization data for the vector bundle E and on the smoothing operators. In the proof of Proposition 3.12, C_n depends also on the Poisson structure π .

3.1. The ingredients of the tame category. We will use some terminology from [10]. A Fréchet space F endowed with an increasing family of semi-norms $\{\|\cdot\|_n\}_{n \geq 0}$ generating its topology will be called a **graded Fréchet space**. A linear map $T : F_1 \rightarrow F_2$ between two graded Fréchet spaces is called **tame** of degree d and base b , if it satisfies inequalities of the form

$$\|Tf\|_n \leq C_n \|f\|_{n+d}, \quad \forall n \geq b, f \in F_1.$$

Let $E \rightarrow S$ be a vector bundle over a compact manifold S and fix a metric on E . For $r > 0$, consider the closed tube in E of radius r

$$E_r := \{v \in E : |v| \leq r\}.$$

The space $\mathfrak{X}^\bullet(E_r)$, of multivector fields on E_r , endowed with the C^n -norms $\|\cdot\|_{n,r}$ is a graded Fréchet space. We recall here the construction of these norms. Fix a finite open cover of S by domains of charts $\{\chi_i : O_i \xrightarrow{\sim} \mathbb{R}^d\}_{i=1}^N$ and vector bundle isomorphisms

$$\tilde{\chi}_i : E|_{O_i} \xrightarrow{\sim} \mathbb{R}^d \times \mathbb{R}^D$$

covering χ_i . We will assume that $\tilde{\chi}_i(E_r|_{O_i}) = \mathbb{R}^d \times \overline{B}_r$ and that the family

$$\{O_i^\delta := \chi_i^{-1}(B_\delta)\}_{i=1}^N$$

covers S for all $\delta \geq 1$. Moreover, we assume that the cover satisfies

$$(7) \quad \text{if } O_i^{3/2} \cap O_j^{3/2} \neq \emptyset \text{ then } O_j^1 \subset O_i^4.$$

This holds if $\chi_i^{-1} : B_4 \rightarrow O_i$ is the exponential corresponding to some metric on S , with injectivity radius bigger than 4.

For $W \in \mathfrak{X}^\bullet(E_r)$, denote its local expression in the chart $\tilde{\chi}_i$ by

$$W_i(z) := \sum_{1 \leq i_1 < \dots < i_p \leq d+D} W_i^{i_1, \dots, i_p}(z) \frac{\partial}{\partial z_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{i_p}},$$

and let the C^n -norm of W be given by

$$\|W\|_{n,r} := \sup_{i, i_1, \dots, i_p} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} W_i^{i_1, \dots, i_p}(z) \right| : z \in B_1 \times B_r, 0 \leq |\alpha| \leq n \right\}.$$

For $s < r$, the restriction maps are norm decreasing

$$\mathfrak{X}^\bullet(E_r) \ni W \mapsto W|_s := W|_{E_s} \in \mathfrak{X}^\bullet(E_s), \quad \|W|_s\|_{n,s} \leq \|W\|_{n,r}.$$

We will work also with the closed subspaces of multivector fields on E_r whose first jet vanishes along S , which we denote by

$$\mathfrak{X}^k(E_r)^{(1)} := \{W \in \mathfrak{X}^k(E_r) : j|_S^1 W = 0\}.$$

The main technical tool used in the Nash-Moser method are the smoothing operators. We will call a family $\{S_t : F \rightarrow F\}_{t>1}$ of linear operators on the graded Fréchet space F **smoothing operators** of degree $d \geq 0$, if there exist constants $C_{n,m} > 0$, such that for all $n, m \geq 0$ and $f \in F$, the following inequalities hold:

$$(8) \quad \|S_t(f)\|_{n+m} \leq t^{m+d} C_{n,m} \|f\|_n, \quad \|S_t(f) - f\|_n \leq t^{-m} C_{n,m} \|f\|_{n+m+d}.$$

The construction of such operators is standard, but since we are dealing with a family of Fréchet spaces $\{\mathfrak{X}^k(E_r)\}_{0 < r \leq 1}$, we give the dependence of $C_{n,m}$ on r .

Lemma 3.1. *The family of graded Fréchet spaces $\{(\mathfrak{X}^k(E_r), \|\cdot\|_{n,r})\}_{r \in (0,1]}$ has a family of smoothing operators of degree $d = 0$*

$$\{S_t^r : \mathfrak{X}^k(E_r) \longrightarrow \mathfrak{X}^k(E_r)\}_{t>1, 0 < r \leq 1},$$

which satisfy (8) with constants of the form $C_{n,m}(r) = C_{n,m} r^{-(n+m+k)}$.

Similarly, the family $\{(\mathfrak{X}^k(E_r)^{(1)}, \|\cdot\|_{n,r})\}_{r \in (0,1]}$ has smoothing operators

$$\{S_t^{r,1} : \mathfrak{X}^k(E_r)^{(1)} \longrightarrow \mathfrak{X}^k(E_r)^{(1)}\}_{t>1, 0 < r \leq 1},$$

of degree $d = 1$ and constants $C_{n,m}(r) = C_{n,m} r^{-(n+m+k+1)}$.

Proof. The existence of smoothing operators of degree zero on the Fréchet space of sections of a vector bundle over a compact manifold (possibly with boundary) is standard (see [10]). We fix such a family $\{S_t : \mathfrak{X}^k(E_1) \rightarrow \mathfrak{X}^k(E_1)\}_{t>1}$. Denote by

$$\mu_\rho : E_R \longrightarrow E_{\rho R}, \quad \mu_\rho(v) := \rho v,$$

the rescaling operators. For $r < 1$, we define S_t^r by conjugation with these operators:

$$S_t^r : \mathfrak{X}^k(E_r) \longrightarrow \mathfrak{X}^k(E_r), \quad S_t^r := \mu_{r^{-1}}^* \circ S_t \circ \mu_r^*.$$

It is straightforward to check that μ_ρ^* satisfies the following inequality

$$\|\mu_\rho^*(W)\|_{n,R} \leq \max\{\rho^{-k}, \rho^n\} \|W\|_{n,\rho R}, \quad \forall W \in \mathfrak{X}^k(E_{\rho R}).$$

Using this, we obtain that S_t^r satisfies (8) with $C_{n,m}(r) = C_{n,m}r^{-(n+m+k)}$.

To construct the operators $S_t^{r,1}$, we first define a tame projection $P : \mathfrak{X}^k(E_r) \rightarrow \mathfrak{X}^k(E_r)^{(1)}$. Choose $\{\lambda_i\}_{i=1}^N$ a smooth partition of unit on S subordinated to the cover $\{O_i^1\}_{i=1}^N$, and let $\{\tilde{\lambda}_i\}_{i=1}^N$ be the pullback to E . For $W \in \mathfrak{X}^k(E_r)$, denote its local representatives by $W_i := \tilde{\chi}_{i,*}(W|_{E_r|_{O_i^1}}) \in \mathfrak{X}^k(\mathbb{R}^d \times \overline{B}_r)$. Define P as follows:

$$P(W) := \sum \tilde{\lambda}_i \cdot \tilde{\chi}_{i,*}^{-1}(W_i - T_y^1(W_i)),$$

where $T_y^1(W_i)$ is the degree one Taylor polynomial of W_i in the fiber direction

$$T_y^1(W_i)(x, y) := W_i(x, 0) + \sum y_j \frac{\partial W_i}{\partial y_j}(x, 0).$$

If $W \in \mathfrak{X}^k(E_r)^{(1)}$, then $T_y^1(W_i) = 0$; so P is a projection. It is easy to check that P is tame of degree 1, that is, there are constants $C_n > 0$ such that

$$\|P(W)\|_{n,r} \leq C_n \|W\|_{n+1,r}.$$

Define the smoothing operators on $\mathfrak{X}^k(E_r)^{(1)}$ as follows:

$$S_t^{r,1} : \mathfrak{X}^k(E_r)^{(1)} \longrightarrow \mathfrak{X}^k(E_r)^{(1)}, \quad S_t^{r,1} := P \circ S_t^r.$$

Using tameness of P , the inequalities for $S_t^{r,1}$ are straightforward. \square

The norms $\|\cdot\|_{n,r}$ satisfy the classical interpolation inequalities with constants which are polynomials in r^{-1} .

Lemma 3.2. *The norms $\|\cdot\|_{n,r}$ satisfy:*

$$\|W\|_{l,r} \leq C_n r^{k-l} (\|W\|_{k,r})^{\frac{n-l}{n-k}} (\|W\|_{n,r})^{\frac{l-k}{n-k}}, \quad \forall r \in (0, 1]$$

for all $0 \leq k \leq l \leq n$, not all equal and all $W \in \mathfrak{X}^\bullet(E_r)$.

Proof. By the interpolation inequalities from [1], it follows that these inequalities hold for the C^n -norms on the spaces $C^\infty(\overline{B}_1 \times \overline{B}_r)$. Applying these to the components of the restrictions to the charts $(E_r|_{O_i^1}, \tilde{\chi}_i)$ of a multivector field in $\mathfrak{X}^\bullet(E_r)$, we obtain the interpolation inequalities on $\mathfrak{X}^\bullet(E_r)$. \square

3.2. Tameness of some natural operators. In this subsection we prove some tameness properties of the Lie bracket, the pullback and the flow of vector fields.

The tame Fréchet Lie algebra of multivector fields. We prove that

$$(\mathfrak{X}^\bullet(E_r), [\cdot, \cdot], \{\|\cdot\|_{n,r}\}_{n \geq 0})$$

forms is a graded tame Fréchet Lie algebra.

Lemma 3.3. *The Schouten bracket on $\mathfrak{X}^\bullet(E_r)$ satisfies*

$$\|[W, V]\|_{n,r} \leq C_n r^{-(n+1)} (\|W\|_{0,r} \|V\|_{n+1,r} + \|W\|_{n+1,r} \|V\|_{0,r}), \quad \forall r \in (0, 1].$$

Proof. By a local computation, the bracket satisfies inequalities of the form:

$$\|[W, V]\|_{n,r} \leq C_n \sum_{i+j=n+1} \|W\|_{i,r} \|V\|_{j,r}.$$

Using the interpolation inequalities, a term in this sum can be bounded by:

$$\|W\|_{i,r} \|V\|_{j,r} \leq C_n r^{-(n+1)} (\|W\|_{0,r} \|V\|_{n+1,r})^{\frac{j}{n+1}} (\|V\|_{0,r} \|W\|_{n+1,r})^{\frac{i}{n+1}}.$$

The following simple inequality (to be used also later) implies the conclusion

$$(9) \quad x^\lambda y^{1-\lambda} \leq x + y, \quad \forall x, y \geq 0, \lambda \in [0, 1].$$

□

The space of local diffeomorphisms. We consider now the space of smooth maps $E_r \rightarrow E$ which are C^1 -close to the inclusion $I_r : E_r \hookrightarrow E$. We call a map $\varphi : E_r \rightarrow E$ **a local diffeomorphism**, if it can be extended on some open to a diffeomorphism onto its image. Since E_r is compact, this is equivalent to injectivity of $d\varphi : TE_r \rightarrow TE$. To be able to measure C^n -norms of such maps, we work with the following open neighborhood of I_r in $C^\infty(E_r; E)$:

$$\mathcal{U}_r := \{\varphi : E_r \rightarrow E : \varphi(E_{r|\overline{O}_i^1}) \subset E_{|O_i}, 1 \leq i \leq N\}.$$

Denote the local representatives of a map $\varphi \in \mathcal{U}_r$ by

$$\varphi_i : \overline{B}_1 \times \overline{B}_r \rightarrow \mathbb{R}^d \times \mathbb{R}^D.$$

Define C^n -distances between maps $\varphi, \psi \in \mathcal{U}_r$ as follows

$$d(\varphi, \psi)_{n,r} := \sup_{1 \leq i \leq N} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} (\varphi_i - \psi_i)(z) \right| : z \in B_1 \times B_r, 0 \leq |\alpha| \leq n \right\}.$$

To control compositions of maps, we will also need the following C^n -distances

$$d(\varphi, \psi)_{n,r,\delta} := \sup_{1 \leq i \leq N} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} (\varphi_i - \psi_i)(z) \right| : z \in B_\delta \times B_r, 0 \leq |\alpha| \leq n \right\},$$

which are well-defined only on the open

$$\mathcal{U}_r^\delta := \{\chi \in \mathcal{U}_r : \chi(E_{r|\overline{O}_i^\delta}) \subset E_{|O_i}\},$$

Similarly, we define also on $\mathfrak{X}^\bullet(E_r)$ norms $\|\cdot\|_{n,r,\delta}$ (these measure the C^n -norms in all our local charts on $B_\delta \times B_r$).

These norms and distances are equivalent.

Lemma 3.4. *There exist $C_n > 0$, such that $\forall r \in (0, 1]$ and $\delta \in [1, 4]$*

$$\begin{aligned} d(\varphi, \psi)_{n,r} &\leq d(\varphi, \psi)_{n,r,\delta} \leq C_n d(\varphi, \psi)_{n,r}, \quad \forall \varphi, \psi \in \mathcal{U}_r^\delta, \\ \|W\|_{n,r} &\leq \|W\|_{n,r,\delta} \leq C_n \|W\|_{n,r}, \quad \forall W \in \mathfrak{X}^\bullet(E_r). \end{aligned}$$

We also use the simplified notations:

$$d(\psi)_{n,r} := d(\psi, I_r)_{n,r}, \quad d(\psi)_{n,r,\delta} := d(\psi, I_r)_{n,r,\delta}.$$

The lemma below is used to check that compositions are defined.

Lemma 3.5. *There exists a constant $\theta > 0$, such that for all $r \in (0, 1]$, $\epsilon \in (0, 1]$, $\delta \in [1, 4]$ and all $\varphi \in \mathcal{U}_r$, satisfying $d(\varphi)_{0,r} < \epsilon\theta$,*

$$\varphi(E_{r|\overline{O}_i^\delta}) \subset E_{r+\epsilon|O_i^{\delta+\epsilon}}.$$

We prove now that I_r has a C^1 -neighborhood of local diffeomorphisms.

Lemma 3.6. *There exists $\theta > 0$, such that, for all $r \in (0, 1]$, if $\psi \in \mathcal{U}_r$ satisfies $d(\psi)_{1,r} < \theta$, then ψ is a local diffeomorphism.*

Proof. By Lemma 3.5, if we shrink θ , we may assume that

$$(10) \quad \psi(E_{r|\overline{O}_i^1}) \subset E_{|O_i^{3/2}}, \quad \psi(E_{r|\overline{O}_i^4}) \subset E_{|O_i}.$$

In a local chart, we write ψ as follows

$$\psi_i := \text{Id} + g_i : \overline{B}_4 \times \overline{B}_r \rightarrow \mathbb{R}^d \times \mathbb{R}^D.$$

By Lemma 3.4, if we shrink θ , we may also assume that

$$(11) \quad \left| \frac{\partial g_i}{\partial z_j}(z) \right| < \frac{1}{2(d+D)}, \quad \forall z \in \overline{B}_4 \times \overline{B}_r.$$

This ensures that $\text{Id} + (dg_i)_z$ is close enough to Id so that it is invertible for all $z \in \overline{B}_1 \times \overline{B}_r$, thus, $(d\psi)_p$ is invertible for all $p \in E_r$.

We check now injectivity of ψ . Let $p^i \in E_{r|O_i^1}$ and $p^j \in E_{r|O_j^1}$ be such that $\psi(p^i) = q = \psi(p^j)$. Then, by (10), $q \in E_{|O_i^{3/2}} \cap E_{|O_j^{3/2}}$, so, by the property (7) of the opens, we know that $O_j^1 \subset O_i^4$, hence $p^i, p^j \in E_{r|O_i^4}$. Denoting by $w^i := \tilde{\chi}_i(p^i)$ and $w^j := \tilde{\chi}_i(p^j)$ we have that $w^i, w^j \in \overline{B}_4 \times \overline{B}_r$. Since $w^i + g_i(w^i) = w^j + g_i(w^j)$, using (11), we obtain

$$\begin{aligned} |w^i - w^j| &= |g_i(w^i) - g_i(w^j)| = \\ &= \left| \int_0^1 \sum_{k=1}^{D+d} \frac{\partial g_i}{\partial z_k} (tw^i + (1-t)w^j) (w_k^i - w_k^j) dt \right| \leq \frac{1}{2} |w^i - w^j|. \end{aligned}$$

Thus $w^i = w^j$, and so $p^i = p^j$. This finishes the proof. \square

The composition satisfies the following tame inequalities.

Lemma 3.7. *There are constants $C_n > 0$ such that for all $1 \leq \delta \leq \sigma \leq 4$ and all $0 < s \leq r \leq 1$, we have that if $\varphi \in \mathcal{U}_s$ and $\psi \in \mathcal{U}_r$ satisfy*

$$\varphi(E_{s|\overline{O_i^s}}) \subset E_{r|O_i^s}, \quad \psi(E_{r|\overline{O_i^s}}) \subset E_{|O_i}, \quad \forall 0 \leq i \leq N,$$

and $d(\varphi)_{1,s} < 1$, then the following inequalities hold:

$$\begin{aligned} d(\psi \circ \varphi)_{n,s,\delta} &\leq d(\psi)_{n,r,\sigma} + d(\varphi)_{n,s,\delta} + \\ &\quad + C_n s^{-n} (d(\psi)_{n,r,\sigma} d(\varphi)_{1,s,\delta} + d(\varphi)_{n,s,\delta} d(\psi)_{1,r,\sigma}), \\ d(\psi \circ \varphi, \psi)_{n,s,\delta} &\leq d(\varphi)_{n,s,\delta} + \\ &\quad + C_n s^{-n} (d(\psi)_{n+1,r,\sigma} d(\varphi)_{1,s,\delta} + d(\varphi)_{n,s,\delta} d(\psi)_{1,r,\sigma}). \end{aligned}$$

Proof. Denote the local expressions of φ and ψ as follows:

$$\begin{aligned} \varphi_i &:= \text{Id} + g_i : \overline{B}_\delta \times \overline{B}_s \longrightarrow B_\sigma \times B_r, \\ \psi_i &:= \text{Id} + f_i : \overline{B}_\sigma \times \overline{B}_r \longrightarrow \mathbb{R}^d \times \mathbb{R}^D. \end{aligned}$$

Then for all $z \in \overline{B}_\delta \times \overline{B}_s$, we can write

$$\psi_i(\varphi_i(z)) - z = f_i(z + g_i(z)) + g_i(z).$$

By computing the $\frac{\partial^{|\alpha|}}{\partial z^\alpha}$ of the right hand side, for a multi-index α with $|\alpha| = n$, we obtain an expression of the form

$$\frac{\partial^{|\alpha|} g_i}{\partial z^\alpha}(z) + \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(\varphi_i(z)) + \sum_{\beta, \gamma_1, \dots, \gamma_p} \frac{\partial^{|\beta|} f_i}{\partial z^\beta}(\varphi_i(z)) \frac{\partial^{|\gamma_1|} g_i^{j_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_i^{j_p}}{\partial z^{\gamma_p}}(z),$$

where the multi-indices in the sum satisfy

$$(12) \quad 1 \leq p \leq n, \quad 1 \leq |\beta|, |\gamma_j| \leq n, \quad |\beta| + \sum_{j=1}^p (|\gamma_j| - 1) = n.$$

The first two terms can be bounded by $d(\psi)_{n,r,\sigma} + d(\varphi)_{n,s,\delta}$. For the last term we use the interpolation inequalities to obtain

$$\begin{aligned} \|f_i\|_{|\beta|,r,\sigma} &\leq C_n s^{1-|\beta|} \|f_i\|_{1,r,\sigma}^{\frac{n-|\beta|}{n-1}} \|f_i\|_{n,r,\sigma}^{\frac{|\beta|-1}{n-1}}, \\ \|g_i\|_{|\gamma_i|,s,\delta} &\leq C_n s^{1-|\gamma_i|} \|g_i\|_{1,s,\delta}^{\frac{n-|\gamma_i|}{n-1}} \|g_i\|_{n,s,\delta}^{\frac{|\gamma_i|-1}{n-1}}. \end{aligned}$$

Multiplying all these, and using (12), the sum is bounded by

$$C_n s^{1-n} \|g_i\|_{1,s,\delta}^{p-1} (\|f_i\|_{1,r,\sigma} \|g_i\|_{n,s,\delta})^{\frac{n-|\beta|}{n-1}} (\|f_i\|_{n,r,\sigma} \|g_i\|_{1,s,\delta})^{\frac{|\beta|-1}{n-1}}.$$

By Lemma 3.4, it follows that $\|g_i\|_{1,s,\delta} < C$, and dropping this term, the first part follows using inequality (9).

For the second part, write for $z \in \overline{B}_\delta \times \overline{B}_s$:

$$\psi_i(\varphi_i(z)) - \psi_i(z) = f_i(z + g_i(z)) - f_i(z) + g_i(z).$$

We compute $\frac{\partial^{|\alpha|}}{\partial z^\alpha}$ of the right hand side, for α a multi-index with $|\alpha| = n$:

$$\begin{aligned} & \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(\varphi_i(z)) - \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(z) + \frac{\partial^{|\alpha|} g_i}{\partial z^\alpha}(z) + \\ & + \sum_{\beta, \gamma_1, \dots, \gamma_p} \frac{\partial^{|\beta|} f_i}{\partial z^\beta}(\varphi_i(z)) \frac{\partial^{|\gamma_1|} g_i^{j_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_i^{j_p}}{\partial z^{\gamma_p}}(z). \end{aligned}$$

where the multi-indices in the sum satisfy (12). The last term we bound as before, and the third by $d(\varphi)_{n,s,\delta}$. Writing the first two terms as

$$\sum_{j=1}^{d+D} \int_0^1 \frac{\partial^{|\alpha|+1} f_i}{\partial z_j \partial z^\alpha}(z + t g_i(z)) g_i^j(z) dt,$$

they are less than $C d(\psi)_{n+1,r,\sigma} d(\varphi)_{0,s,\delta}$. Adding up, the result follows. \square

We give now conditions for infinite compositions of maps to converge.

Lemma 3.8. *There exists $\theta > 0$, such that for all sequences*

$$\{\varphi_k \in \mathcal{U}_{r_k}\}_{k \geq 1}, \quad \varphi_k : E_{r_k} \longrightarrow E_{r_{k-1}},$$

where $0 < r < r_k < r_{k-1} \leq r_0 < 1$, which satisfy

$$\sigma_0 := \sum_{k \geq 1} d(\varphi_k)_{0,r_k} < \theta, \quad \sigma_n := \sum_{k \geq 1} d(\varphi_k)_{n,r_k} < \infty, \quad \forall n \geq 1,$$

the sequence of maps

$$\psi_k := \varphi_1 \circ \dots \circ \varphi_k : E_{r_k} \longrightarrow E_{r_0},$$

converges in all C^n -norms on E_r to a map $\psi : E_r \rightarrow E_{r_0}$, with $\psi \in \mathcal{U}_r$. Moreover, there are $C_n > 0$, such that if $d(\varphi_k)_{1,r_k} < 1$, $\forall k \geq 1$, then

$$d(\psi)_{n,r} \leq e^{C_n r^{-n} \sigma_n} C_n r^{-n} \sigma_n.$$

Proof. Consider the following sequences of numbers:

$$\epsilon_k := \frac{d(\varphi_k)_{0,r_k}}{\sum_{l \geq 1} d(\varphi_l)_{0,r_l}}, \quad \delta_k := 2 - \sum_{l=1}^k \epsilon_l.$$

We have that $d(\varphi_k)_{0,r_k} \leq \epsilon_k \theta$. So, by Lemma 3.5, we may assume that

$$\varphi_k(E_{r_k} | \overline{O}_i^2) \subset E_{r_{k-1} | O_i}, \quad \varphi_k(E_{r_k} | \overline{O}_i^{\delta_k}) \subset E_{r_{k-1} | O_i^{\delta_{k-1}}},$$

and this implies that

$$\psi_{k-1}(E_{r_{k-1} | \overline{O}_i^{\delta_{k-1}}}) \subset E_{r_0 | O_i}.$$

So we can apply Lemma 3.7 to the pair ψ_{k-1} and φ_k for all $k > k_0$. The first part of Lemma 3.7 and Lemma 3.4 imply an inequality of the form:

$$1 + d(\psi_k)_{n,r_k,\delta_k} \leq (1 + d(\psi_{k-1})_{n,r_{k-1},\delta_{k-1}})(1 + C_n r^{-n} d(\varphi_k)_{n,r_k}).$$

Iterating this inequality, we obtain that

$$\begin{aligned} 1 + d(\psi_k)_{n,r_k,\delta_k} &\leq (1 + d(\psi_{k_0})_{n,r_{k_0},\delta_{k_0}}) \prod_{l=k_0+1}^k (1 + C_n r^{-n} d(\varphi_l)_{n,r_l}) \leq \\ &\leq (1 + d(\psi_{k_0})_{n,r_{k_0},\delta_{k_0}}) e^{C_n r^{-n} \sum_{l>k_0} d(\varphi_l)_{n,r_l}} \leq \\ &\leq (1 + d(\psi_{k_0})_{n,r_{k_0},\delta_{k_0}}) e^{C_n r^{-n} \sigma_n}. \end{aligned}$$

The second part of Lemma 3.7 and Lemma 3.4 imply

$$\begin{aligned} d(\psi_k, \psi_{k-1})_{n,r} &\leq (1 + d(\psi_{k-1})_{n+1,r_{k-1},\delta_{k-1}}) C_n r^{-n} d(\varphi_k)_{n,r_k,\delta_k} \leq \\ &\leq (1 + d(\psi_{k_0})_{n+1,r_{k_0},\delta_{k_0}}) e^{C_{n+1} r^{-1-n} \sigma_{n+1}} C_n r^{-n} d(\varphi_k)_{n,r_k}. \end{aligned}$$

This shows that the sum $\sum_{k \geq 1} d(\psi_k, \psi_{k-1})_{n,r}$ converges for all n , hence the sequence $\{\psi_k|_{E_r}\}_{k \geq 1}$ converges in all C^n -norms to a smooth function $\psi : E_r \rightarrow E_{r_0}$.

If $d(\varphi_k)_{1,r_k} < 1$ for all $k \geq 1$, then we can take $k_0 = 0$. So, we obtain

$$1 + d(\psi_k)_{n,r_k,\delta_k} \leq \prod_{l=1}^k (1 + C_n r^{-n} d(\varphi_l)_{n,r_l}) \leq e^{C_n r^{-n} \sum_{l=1}^k d(\varphi_l)_{n,r_l}} \leq e^{C_n r^{-n} \sigma_n}.$$

Using the trivial inequality $e^x - 1 \leq x e^x$, for $x \geq 0$, the result follows. \square

Tameness of the flow. The C^0 -norm of a vector field controls the size of the domain of its flow.

Lemma 3.9. *There exists $\theta > 0$ such that for all $0 < s < r \leq 1$ and all $X \in \mathfrak{X}^1(E_r)$ with $\|X\|_{0,r} < (r-s)\theta$, we have that φ_X^t , the flow of X , is defined for all $t \in [0, 1]$ on E_s and belongs to \mathcal{U}_s .*

Proof. We denote the restriction of X to a chart by $X_i \in \mathfrak{X}^1(\mathbb{R}^d \times \overline{B}_r)$. Consider $p \in \overline{B}_1 \times \overline{B}_s$. Let $t \in (0, 1]$ be such that the flow of X_i is defined up to time t at p and such that for all $\tau \in [0, t)$ it satisfies $\varphi_{X_i}^\tau(p) \in B_2 \times B_r$. Then we have that

$$|\varphi_{X_i}^t(p) - p| = \left| \int_0^t d(\varphi_{X_i}^\tau(p)) \right| \leq \int_0^t |X_i(\varphi_{X_i}^\tau(p))| d\tau \leq \|X_i\|_{0,r,2} \leq C \|X\|_{0,r},$$

where for the last step we used Lemma 3.4. Hence, if $\|X\|_{0,r} < (r-s)/C$, we have that $\varphi_{X_i}^t(p) \in B_2 \times B_r$, and this implies the result. \square

We prove now that the map which associates to a vector field its flow is tame (this proof was inspired by the proof of Lemma B.3 in [15]).

Lemma 3.10. *There exists $\theta > 0$ such that for all $0 < s < r \leq 1$, and all $X \in \mathfrak{X}^1(E_r)$ with*

$$\|X\|_{0,r} < (r-s)\theta, \quad \|X\|_{1,r} < \theta$$

we have that $\varphi_X := \varphi_X^1$ belongs to \mathcal{U}_s and it satisfies:

$$d(\varphi_X)_{0,s} \leq C_0 \|X\|_{0,r}, \quad d(\varphi_X)_{n,s} \leq r^{1-n} C_n \|X\|_{n,r}, \quad \forall n \geq 1.$$

Proof. By Lemma 3.9, for $t \in [0, 1]$, we have that $\varphi_X^t \in \mathcal{U}_s$, and by its proof that the local representatives take values in $B_2 \times B_r$

$$\varphi_{X_i}^t := \text{Id} + g_{i,t} : \overline{B}_1 \times \overline{B}_s \longrightarrow B_2 \times B_r.$$

We will prove by induction on n that $g_{i,t}$ satisfies inequalities of the form:

$$(13) \quad \|g_{i,t}\|_{n,s} \leq C_n P_n(X),$$

where $P_n(X)$ denotes the following polynomials in the norms of X

$$P_0(X) = \|X\|_{0,r}, \quad P_1(X) = \|X\|_{1,r},$$

$$P_n(X) = \sum_{\substack{j_1 + \dots + j_p = n-1 \\ 1 \leq j_k \leq n-1}} \|X\|_{j_1+1,r} \dots \|X\|_{j_p+1,r}.$$

Observe that (13) implies the conclusion, since by the interpolation inequalities and the fact that $\|X\|_{1,r} < \theta \leq 1$ we have that

$$\|X\|_{j_k+1,r} \leq C_n r^{-j_k} (\|X\|_{1,r})^{1-\frac{j_k}{n-1}} (\|X\|_{n,r})^{\frac{j_k}{n-1}} \leq C_n r^{-j_k} \|X\|_{n,r}^{\frac{j_k}{n-1}},$$

hence

$$P_n(X) \leq C_n r^{1-n} \|X\|_{n,r}.$$

The map $g_{i,t}$ satisfies the ordinary differential equation

$$\frac{dg_{i,t}}{dt}(z) = \frac{d\varphi_{X_i}^t}{dt}(z) = X_i(\varphi_{X_i}^t(z)) = X_i(g_{i,t}(z) + z).$$

Since $g_{i,0} = 0$, it follows that

$$(14) \quad g_{i,t}(z) = \int_0^t X_i(z + g_{i,\tau}(z)) d\tau.$$

Using also Lemma 3.4, we obtain the result for $n = 0$:

$$\|g_{i,t}\|_{0,s} \leq \|X\|_{0,r,2} \leq C_0 \|X\|_{0,r}.$$

We will use the following version of the Gronwall inequality: if $u : [0, 1] \rightarrow \mathbb{R}$ is a continuous map and there are positive constants A, B such that

$$u(t) \leq A + B \int_0^t u(\tau) d\tau,$$

then u satisfies $u(t) \leq Ae^B$.

Computing the partial derivative $\frac{\partial}{\partial z_j}$ of equation (14) we obtain

$$\frac{\partial g_{i,t}}{\partial z_j}(z) = \int_0^t \left(\frac{\partial X_i}{\partial z_j}(z + g_{i,\tau}(z)) + \sum_{k=1}^{D+d} \frac{\partial X_i}{\partial z_k}(z + g_{i,\tau}(z)) \frac{\partial g_{i,\tau}^k}{\partial z_j}(z) \right) d\tau.$$

Therefore, using again Lemma 3.4, the function $|\frac{\partial g_{i,t}}{\partial z_j}(z)|$ satisfies:

$$|\frac{\partial g_{i,t}}{\partial z_j}(z)| \leq C \|X\|_{1,r} + (D+d) \|X\|_{1,r} \int_0^t |\frac{\partial g_{i,\tau}}{\partial z_j}(z)| d\tau.$$

The case $n = 1$ follows now by Gronwall's inequality:

$$\|\frac{\partial g_{i,t}}{\partial z_j}\|_{0,s} \leq C \|X\|_{1,r} e^{(D+d)\|X\|_{1,r}} \leq C \|X\|_{1,r}.$$

For a multi-index α , with $|\alpha| = n \geq 2$, applying $\frac{\partial^{|\alpha|}}{\partial z^\alpha}$ to (14), we obtain

$$(15) \quad \frac{\partial^{|\alpha|} g_{i,t}}{\partial z^\alpha}(z) = \int_0^t \sum_{2 \leq |\beta| \leq |\alpha|} \frac{\partial^{|\beta|} X_i}{\partial z^\beta}(z + g_{i,\tau}(z)) \frac{\partial^{|\gamma_1|} g_{i,\tau}^{i_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_{i,\tau}^{i_p}}{\partial z^{\gamma_p}}(z) d\tau + \\ + \int_0^t \sum_{j=1}^{D+d} \frac{\partial X_i}{\partial z_j}(z + g_{i,\tau}(z)) \frac{\partial^{|\alpha|} g_{i,\tau}^j}{\partial z^\alpha}(z) d\tau,$$

where the multi-indices satisfy

$$1 \leq |\gamma_k| \leq n-1, \quad (|\gamma_1| - 1) + \dots + (|\gamma_p| - 1) + |\beta| = n.$$

Since $|\gamma_k| \leq n-1$, we can apply induction to conclude that

$$\|\frac{\partial^{|\gamma_k|} g_{i,\tau}^{i_k}}{\partial z^{\gamma_k}}\|_{0,s} \leq P_{|\gamma_k|}(X).$$

So, the first part of the sum can be bounded by

$$(16) \quad C_n \sum_{\substack{j_0+\dots+j_p=n-1 \\ 1 \leq j_k \leq n-1}} \|X\|_{j_0+1,1} P_{j_1+1}(X) \dots P_{j_p+1}(X).$$

It is easy to see that the polynomials $P_k(X)$ satisfy:

$$(17) \quad P_{u+1}(X)P_{v+1}(X) \leq C_{u,v}P_{u+v+1}(X),$$

therefore (16) can be bounded by $C_n P_n(X)$. Using this in (15), we obtain

$$\left| \frac{\partial^{|\alpha|} g_{i,t}}{\partial z^\alpha}(z) \right| \leq C_n P_n(X) + (D+d) \|X\|_{1,r} \int_0^t \left| \frac{\partial^{|\alpha|} g_{i,\tau}}{\partial z^\alpha}(z) \right| d\tau.$$

Applying Gronwall's inequality, we obtain the conclusion. \square

We show now how to approximate pullbacks by flows of vector fields.

Lemma 3.11. *There exists $\theta > 0$, such that for all $0 < s < r \leq 1$ and all $X \in \mathfrak{X}^1(E_r)$ with $\|X\|_{0,r} < (r-s)\theta$ and $\|X\|_{1,r} < \theta$, we have that*

$$\begin{aligned} \|\varphi_X^*(W)\|_{n,s} &\leq C_n r^{-n} (\|W\|_{n,r} + \|W\|_{0,r} \|X\|_{n+1,r}), \\ \|\varphi_X^*(W) - W|_s\|_{n,s} &\leq C_n r^{-2n-1} (\|X\|_{n+1,r} \|W\|_{1,r} + \|X\|_{1,r} \|W\|_{n+1,r}), \\ \|\varphi_X^*(W) - W|_s - \varphi_X^*([X, W])\|_{n,s} &\leq \\ &\leq C_n r^{-3(n+2)} \|X\|_{0,r} (\|X\|_{n+2,r} \|W\|_{2,r} + \|X\|_{2,r} \|W\|_{n+2,r}), \end{aligned}$$

for all $W \in \mathfrak{X}^\bullet(E_r)$, where $C_n > 0$ is a constant depending only on n .

Proof. As in the proof above, the local expression of φ_X is defined as follows:

$$\varphi_{X_i} = \text{Id} + g_i : \overline{B}_1 \times \overline{B}_s \longrightarrow B_2 \times B_r.$$

Let $W \in \mathfrak{X}^\bullet(E_r)$, and denote by W_i its local expression on $E_r|_{\overline{O}_2^i}$:

$$W_i := \sum_{J=\{j_1 < \dots < j_k\}} W_i^J(z) \frac{\partial}{\partial z_{j_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{j_k}} \in \mathfrak{X}^\bullet(\overline{B}_2 \times \overline{B}_r).$$

The local representative of $\varphi_X^*(W)$, is given for $z \in \overline{B}_1 \times \overline{B}_s$ by

$$(\varphi_X^* W)_i = \sum_J W_i^J(z + g_i(z)) (\text{Id} + d_z g_i)^{-1} \frac{\partial}{\partial z_{j_1}} \wedge \dots \wedge (\text{Id} + d_z g_i)^{-1} \frac{\partial}{\partial z_{j_k}}.$$

By the Cramer rule, the matrix $(\text{Id} + d_z g_i)^{-1}$ has entries of the form

$$\Psi \left(\frac{\partial g_i^l}{\partial z_j}(z) \right) \det(\text{Id} + d_z g_i)^{-1},$$

where Ψ is a polynomial in the variables Y_j^l , which we substitute by $\frac{\partial g_i^l}{\partial z_j}(z)$. Therefore, any coefficient of the local expression of $\varphi_X^*(W)_i$, will be a sum of elements of the form

$$W_i^J(z + g_i(z)) \Psi \left(\frac{\partial g_i^l}{\partial z_j}(z) \right) \det(\text{Id} + d_z g_i)^{-k}.$$

When computing $\frac{\partial^{|\alpha|}}{\partial z^\alpha}$ of such an expression, with $|\alpha| = n$, using an inductive argument, one proves that the outcome is a sum of terms of the form

$$(18) \quad \frac{\partial^{|\beta|} W_i^J}{\partial z^\beta}(z + g_i(z)) \frac{\partial^{|\gamma_1|} g_i^{v_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_i^{v_p}}{\partial z^{\gamma_p}}(z) \det(\text{Id} + d_z g_i)^{-M},$$

with coefficients depending only on α and on the multi-indices, which satisfy

$$0 \leq p, \quad 0 \leq M, \quad 1 \leq |\gamma_j|, \quad |\beta| + (|\gamma_1| - 1) + \dots + (|\gamma_p| - 1) = n.$$

By Lemma 3.10, $\|g_i\|_{1,s} < C\theta$, so, if we shrink θ , we find that

$$\det(\text{Id} + d_z g_i)^{-1} < 2, \quad \forall z \in \overline{B}_1 \times \overline{B}_s.$$

Using this, Lemma 3.4 for W and $|\frac{\partial g_i}{\partial z_j}(z)| \leq C$, we bound (18) by

$$C_n \sum_{j, j_1, \dots, j_p} \|W\|_{j,r} \|g_i\|_{j_1+1,s} \cdots \|g_i\|_{j_p+1,s},$$

where the indexes satisfy

$$0 \leq j, \quad 0 \leq j_k, \quad j + j_1 + \dots + j_p = n.$$

The term with $p = 0$ can be simply bounded by $C_n \|W\|_{n,r}$. For the other terms, we will use the bound $\|g_i\|_{j_k+1,s} \leq P_{j_k+1}(X)$ from the proof of Lemma 3.10. The multiplicative property (17) of the polynomials $P_l(X)$ implies

$$\|\varphi_X^*(W)\|_{n,s} \leq C_n \sum_{j=0}^n \|W\|_{j,r} P_{n-j+1}(X).$$

Applying interpolation to $W_{j,r}$ and to a term of $P_{n-j+1}(X)$ we obtain

$$\begin{aligned} \|W\|_{j,r} &\leq C_n r^{-j} \|W\|_{0,r}^{1-j/n} \|W\|_{n,r}^{j/n}, \\ \|X\|_{j_k+1,r} &\leq C_n r^{-j_k} \|X\|_{1,r}^{1-j_k/n} \|X\|_{n+1,r}^{j_k/n} \leq C_n r^{-j_k} \|X\|_{n+1,r}^{j_k/n}. \end{aligned}$$

Multiplying all these terms and using (9), conclude the first part of the proof

$$\begin{aligned} \|W\|_{j,r} \|X\|_{j_1+1,r} \cdots \|X\|_{j_p+1,r} &\leq C_n r^{-n} (\|W\|_{0,r} \|X\|_{n+1,r})^{1-j/n} \|W\|_{n,r}^{j/n} \\ &\leq C_n r^{-n} (\|W\|_{n,r} + \|W\|_{0,r} \|X\|_{n+1,r}). \end{aligned}$$

For the second inequality, denote by

$$W_t := \varphi_X^{t*}(W) - W|_s \in \mathfrak{X}^\bullet(E_s).$$

Then $W_0 = 0$, $W_1 = \varphi_X^*(W) - W|_s$ and $\frac{d}{dt} W_t = \varphi_X^{t*}([X, W])$, therefore

$$\varphi_X^*(W) - W|_s = \int_0^1 \varphi_X^{t*}([X, W]) dt.$$

By the first part, we obtain

$$\|\varphi_X^*(W) - W|_s\|_{n,s} \leq C_n r^{-n} (\|[X, W]\|_{n,r} + \|[X, W]\|_{0,r} \|X\|_{n+1,r}).$$

Using now Lemma 3.3 and that $\|X\|_{1,r} \leq \theta$ we obtain the second part:

$$\|\varphi_X^*(W) - W|_s\|_{n,s} \leq C_n r^{-2n-1} (\|X\|_{n+1,r} \|W\|_{1,r} + \|W\|_{1,r} \|X\|_{n+1,r}).$$

For the last inequality, denote by

$$W_t := \varphi_X^{t*}(W) - W|_s - t\varphi_X^{t*}([X, W]).$$

Then we have that $W_0 = 0$ and $W_1 = \varphi_X^*(W) - W|_s - \varphi_X^*([X, W])$ and

$$\frac{d}{dt} W_t = -t\varphi_X^{t*}([X, [X, W]]),$$

therefore

$$W_1 = - \int_0^1 t\varphi_X^{t*}([X, [X, W]]) dt.$$

Using again the first part, it follows that

$$(19) \quad \|W_1\|_{n,s} \leq C_n r^{-n} (\|[X, [X, W]]\|_{n,r} + \|[X, [X, W]]\|_{0,r} \|X\|_{n+1,r}).$$

Applying twice Lemma 3.3, for all $k \leq n$ we obtain that:

$$\begin{aligned} \|[X, [X, W]]\|_{k,r} &\leq C_n(r^{-(k+3)}\|X\|_{k+1,r}(\|X\|_{0,r}\|W\|_{1,r} + \|X\|_{1,r}\|W\|_{0,r}) + \\ &\quad + r^{-(2k+3)}\|X\|_{0,r}(\|X\|_{0,r}\|W\|_{k+2,r} + \|X\|_{k+2,r}\|W\|_{0,r})) \leq \\ &\leq C_n r^{-(2k+5)}\|X\|_{0,r}(\|W\|_{k+2,r}\|X\|_{0,r} + \|W\|_{2,r}\|X\|_{k+2,r}), \end{aligned}$$

where we have used the interpolation inequality

$$\|X\|_{1,r}\|X\|_{k+1,r} \leq C_n r^{-(k+2)}\|X\|_{0,r}\|X\|_{k+2,r}.$$

The first term in (19) can be bounded using this inequality for $k = n$. For $k = 0$, using also that $\|X\|_{1,r} \leq \theta$ and the interpolation inequality

$$\|X\|_{2,r}\|X\|_{n+1,r} \leq C_n r^{-(n+1)}\|X\|_{1,r}\|X\|_{n+2,r},$$

we can bound the second term in (19), and this concludes the proof:

$$\|[X, [X, W]]\|_{0,r}\|X\|_{n+1,r} \leq C_n r^{-(n+6)}\|W\|_{2,r}\|X\|_{0,r}\|X\|_{n+2,r}.$$

□

3.3. An invariant tubular neighborhood and tame homotopy operators.

We start now the proof of Theorem 2. We will use two results presented in the appendix: existence of invariant tubular neighborhood (Lemma A.1) and the Tame Vanishing Lemma (Lemma C.1).

Let (M, π) and $S \subset M$ be as in the statement. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid integrating T^*M . By restricting to the connected components of the identities in the s -fibers of \mathcal{G} [16], we may assume that \mathcal{G} has connected s -fibers.

By Lemma A.1, S has an invariant tubular neighborhood $E \cong \nu_S$ endowed with a metric, such that the closed tubes $E_r := \{v \in E \mid |v| \leq r\}$, for $r > 0$, are also \mathcal{G} -invariant. We endow E with all the structure from subsection 3.1.

Since E is invariant, the cotangent Lie algebroid of (E, π) is integrable by $\mathcal{G}|_E$, which has compact s -fibers with vanishing H^2 . Therefore, by the Tame Vanishing Lemma and Corollaries C.2, C.3 from the appendix, and there are linear homotopy operators

$$\begin{aligned} \mathfrak{X}^1(E) &\xleftarrow{h_1} \mathfrak{X}^2(E) \xleftarrow{h_2} \mathfrak{X}^3(E), \\ [\pi, h_1(V)] + h_2([\pi, V]) &= V, \quad \forall V \in \mathfrak{X}^2(E), \end{aligned}$$

which satisfy:

- they induce linear homotopy operators h_1^r and h_2^r on $(E_r, \pi|_r)$;
- there are constants $C_n > 0$ such that, for all $r \in (0, 1]$,

$$\|h_1^r(X)\|_{n,r} \leq C_n \|X\|_{n+s,r}, \quad \|h_2^r(Y)\|_{n,r} \leq C_n \|Y\|_{n+s,r},$$

for all $X \in \mathfrak{X}^2(E_r)$, $Y \in \mathfrak{X}^3(E_r)$, where $s = \lfloor \frac{1}{2} \dim(M) \rfloor + 1$;

- they induce homotopy operators in second degree on the subcomplex of vector fields vanishing along S .

3.4. The Nash-Moser method. We fix radii $0 < r < R < 1$. Let s be as in the previous subsection, and let

$$\alpha := 2(s+5), \quad p := 7(s+4).$$

Then p is the integer from the statement of Theorem 2. Consider $\tilde{\pi}$ a second Poisson structure defined on E_R . To $\tilde{\pi}$ we associate the inductive procedure:

Procedure \mathbf{P}_0 : Consider

- the number

$$t(\tilde{\pi}) := \|\pi - \tilde{\pi}\|_{p,R}^{-1/\alpha},$$

- the sequences of numbers

$$\begin{aligned} \epsilon_0 &:= (R - r)/4, & r_0 &:= R, & t_0 &:= t(\tilde{\pi}), \\ \epsilon_{k+1} &:= \epsilon_k^{3/2}, & r_{k+1} &:= r_k - \epsilon_k, & t_{k+1} &:= t_k^{3/2}, \end{aligned}$$

- the sequences of Poisson bivectors and vector fields

$$\{\pi_k \in \mathfrak{X}^2(E_{r_k})\}_{k \geq 0}, \quad \{X_k \in \mathfrak{X}^1(E_{r_k})\}_{k \geq 0},$$

defined inductively by

$$(20) \quad \pi_0 := \tilde{\pi}, \quad \pi_{k+1} := \varphi_{X_k}^*(\pi_k), \quad X_k := S_{t_k}^{r_k}(h_1^{r_k}(\pi_k - \pi|_{r_k})),$$

- the sequence of maps

$$\psi_k := \varphi_{X_0} \circ \dots \circ \varphi_{X_k} : E_{r_{k+1}} \longrightarrow E_R.$$

By our choice of ϵ_0 , observe that $r < r_k < R$ for all $k \geq 1$:

$$\sum_{k=0}^{\infty} \epsilon_k = \sum_{k=0}^{\infty} \epsilon_0^{(3/2)^k} < \sum_{k=0}^{\infty} \epsilon_0^{1+\frac{k}{2}} = \frac{\epsilon_0}{1 - \sqrt{\epsilon_0}} \leq (R - r),$$

For **Procedure \mathbf{P}_0** to be well-defined, we need that

(C_k) the time one flow of X_k is defined as a map between

$$\varphi_{X_k} : E_{r_{k+1}} \longrightarrow E_{r_k}.$$

For part (b) of Theorem 2, we consider also the **Procedure \mathbf{P}_1** , associated to $\tilde{\pi}$ such that $j_{|S}^1 \tilde{\pi} = j_{|S}^1 \pi$. We define **Procedure \mathbf{P}_1** the same as **Procedure \mathbf{P}_0** , except that in (20) we use the smoothing operators $S_{t_k}^{r_k, 1}$.

To show that **Procedure \mathbf{P}_1** is well-defined, besides for condition (C_k), one should also check that $h_1^{r_k}(\pi_k - \pi|_{r_k}) \in \mathfrak{X}^1(E_{r_k})^{(1)}$. Nevertheless, this always holds by the property of $h_1^{r_k}$, that it preserves the space of tensors vanishing up to first order, and the fact that $j_{|S}^1(\pi_k - \pi|_{r_k}) = 0$. This last claim is proved inductively: By hypothesis, $j_{|S}^1(\pi_0 - \pi|_R) = 0$. Assume that $j_{|S}^1(\pi_k - \pi|_{r_k}) = 0$, for some $k \geq 0$. Then, as before, also $X_k \in \mathfrak{X}^1(E_{r_k})^{(1)}$, hence the first order jet of φ_{X_k} along S is that of the identity, and so

$$j_{|S}^1(\pi_{k+1}) = j_{|S}^1(\pi_k) = j_{|S}^1(\pi).$$

Therefore $j_{|S}^1(\pi_{k+1} - \pi|_{r_{k+1}}) = 0$.

Procedure \mathbf{P}_0 produces the map ψ from Theorem 2.

Proposition 3.12. *There exists $\delta > 0$ and an integer $d \geq 0$, such that both procedures \mathbf{P}_0 and \mathbf{P}_1 are well defined for every $\tilde{\pi}$ satisfying*

$$(21) \quad \|\tilde{\pi} - \pi\|_{p,R} < \delta(r(R - r))^d.$$

Moreover, the sequence $\psi_{k|_r}$ converges uniformly on E_r with all its derivatives to a local diffeomorphism ψ , which is a Poisson map between

$$\psi : (E_r, \pi|_r) \longrightarrow (E_R, \tilde{\pi}),$$

and which satisfies

$$(22) \quad d(\psi)_{1,r} \leq \|\pi - \tilde{\pi}\|_{p,R}^{1/\alpha}.$$

*If $j_{|S}^1 \tilde{\pi} = j_{|S}^1 \pi$, then ψ obtained by **Procedure \mathbf{P}_1** is the identity along S up to first order.*

Proof. We will prove the statement for the two procedures simultaneously. We denote by S_k the used smoothing operators, that is, in \mathbf{P}_0 we let $S_k := S_{t_k}^{r_k}$ and in \mathbf{P}_1 we let $S_k := S_{t_k}^{r_k,1}$. In both cases, these satisfy the inequalities:

$$\begin{aligned} \|S_k(X)\|_{m,r_k} &\leq C_m r^{-c_m} t_k^{l+1} \|X\|_{m-l,r_k}, \\ \|S_k(X) - X\|_{m-l,r_k} &\leq C_m r^{-c_m} t_k^{-l} \|X\|_{m+1,r_k}. \end{aligned}$$

For the procedures to be well-defined and to converge, we need that $t_0 = t(\tilde{\pi})$ is big enough, more precisely it will have to satisfy a finite number of inequalities of the form

$$(23) \quad t_0 = t(\tilde{\pi}) > C(r(R-r))^{-c}.$$

Taking $\tilde{\pi}$ such that it satisfies (21), it suffices to ask that δ is small enough and d is big enough, such that a finite number of inequalities of the form

$$\delta((R-r)r)^d < \frac{1}{C}(r(R-r))^c$$

hold, and then t_0 will satisfy (23).

Also, since $t_0 > 4(R-r)^{-1} = \epsilon_0^{-1}$, it follows that

$$t_k > \epsilon_k^{-1}, \quad \forall k \geq 0.$$

We will prove inductively that the bivectors

$$Z_k := \pi_k - \pi|_{r_k} \in \mathfrak{X}^2(E_{r_k})$$

satisfy the inequalities (a_k) and (b_k)

$$(a_k) \quad \|Z_k\|_{s,r_k} \leq t_k^{-\alpha}, \quad (b_k) \quad \|Z_k\|_{p,r_k} \leq t_k^\alpha.$$

Since $t_0^{-\alpha} = \|Z_0\|_{p,R}$, (a_0) and (b_0) hold. Assuming that (a_k) and (b_k) hold for some $k \geq 0$, we will show that condition (C_k) holds (i.e. the procedure is well-defined up to step k) and also that (a_{k+1}) and (b_{k+1}) hold.

First we give a bound for the norms of X_k in terms of the norms of Z_k

$$(24) \quad \begin{aligned} \|X_k\|_{m,r_k} &= \|S_k(h_1^{r_k}(Z_k))\|_{m,r_k} \leq C_m r^{-c_m} t_k^{1+l} \|h_1^{r_k}(Z_k)\|_{m-l,r_k} \leq \\ &\leq C_m r^{-c_m} t_k^{1+l} \|Z_k\|_{m+s-l,r_k}, \quad \forall \quad 0 \leq l \leq m. \end{aligned}$$

In particular, for $m = l$, we obtain

$$(25) \quad \|X_k\|_{m,r_k} \leq C_m r^{-c_m} t_k^{1+m-\alpha}.$$

Since $\alpha > 4$ and $t_k > \epsilon_k^{-1}$, this inequality implies that

$$(26) \quad \|X_k\|_{1,r_k} \leq C r^{-c} t_k^{2-\alpha} \leq C r^{-c} t_0^{-1} t_k^{-1} < C r^{-c} t_0^{-1} \epsilon_k.$$

Since $t_0 > C r^{-c} / \theta$, we have that $\|X_k\|_{1,r_k} \leq \theta \epsilon_k$, and so by Lemma 3.9 (C_k) holds. Moreover, X_k satisfies the inequalities from Lemma 3.10 and Lemma 3.11.

We deduce now an inequality for all norms $\|Z_{k+1}\|_{n,r_{k+1}}$, with $n \geq s$

$$(27) \quad \begin{aligned} \|Z_{k+1}\|_{n,r_{k+1}} &= \|\varphi_{X_k}^*(Z_k) + \varphi_{X_k}^*(\pi) - \pi\|_{n,r_{k+1}} \leq \\ &\leq C_n r^{-c_n} (\|Z_k\|_{n,r_k} + \|X_k\|_{n+1,r_k} \|Z_k\|_{0,r_k} + \|X_k\|_{n+1,r_k} \|\pi\|_{n+1,r_k}) \leq \\ &\leq C_n r^{-c_n} (\|Z_k\|_{n,r_k} + \|X_k\|_{n+1,r_k}) \leq C_n r^{-c_n} t_k^{s+2} \|Z_k\|_{n,r_k}, \end{aligned}$$

where we used Lemma 3.11, the inductive hypothesis and inequality (24) with $m = n+1$ and $l = s+1$. For $n = p$, using also that $s+2+\alpha \leq \frac{3}{2}\alpha - 1$, this gives (b_{k+1}) :

$$\|Z_{k+1}\|_{p,r_{k+1}} \leq C r^{-c} t_k^{s+2+\alpha} \leq C r^{-c} t_k^{\frac{3}{2}\alpha-1} \leq C r^{-c} t_0^{-1} t_{k+1}^\alpha \leq t_{k+1}^\alpha.$$

To prove (a_{k+1}) , we write $Z_{k+1} = V_k + \varphi_{X_k}^*(U_k)$, where

$$V_k := \varphi_{X_k}^*(\pi) - \pi - \varphi_{X_k}^*([X_k, \pi]), \quad U_k := Z_k - [\pi, X_k].$$

Using Lemma 3.11 and inequality (25), we bound the two terms by

$$(28) \quad \|V_k\|_{s,r_{k+1}} \leq Cr^{-c} \|\pi\|_{s+2,r_k} \|X_k\|_{0,r_k} \|X_k\|_{s+2,r_k} \leq Cr^{-c} t_k^{s+4-2\alpha},$$

$$(29) \quad \|\varphi_{X_k}^*(U_k)\|_{s,r_{k+1}} \leq Cr^{-c} (\|U_k\|_{s,r_k} + \|U_k\|_{0,r_k} \|X_k\|_{s+1,r_k}) \leq \\ \leq Cr^{-c} (\|U_k\|_{s,r_k} + t_k^{s+2-\alpha} \|U_k\|_{0,r_k}).$$

To compute the C^s -norm for U_k , we rewrite it as

$$U_k = Z_k - [\pi, X_k] = [\pi, h_1^{r_k}(Z_k)] + h_2^{r_k}([\pi, Z_k]) - [\pi, X_k] = \\ = [\pi, (I - S_k)h_1^{r_k}(Z_k)] - \frac{1}{2}h_2^{r_k}([Z_k, Z_k]).$$

By tameness of the Lie bracket, the first term can be bounded by

$$\|[\pi, (I - S_k)h_1^{r_k}(Z_k)]\|_{s,r_k} \leq Cr^{-c} \|(I - S_k)h_1^{r_k}(Z_k)\|_{s+1,r_k} \leq \\ \leq Cr^{-c} t_k^{2-p+2s} \|h_1^{r_k}(Z_k)\|_{p-s,r_k} \leq Cr^{-c} t_k^{2-p+2s} \|Z_k\|_{p,r_k} \leq \\ \leq Cr^{-c} t_k^{2-p+2s+\alpha} = Cr^{-c} t_k^{-\frac{3}{2}\alpha-1},$$

and using also the interpolation inequalities, for the second term we obtain

$$\|\frac{1}{2}h_2^{r_k}([Z_k, Z_k])\|_{s,r_k} \leq C\|[Z_k, Z_k]\|_{2s,r_k} \leq Cr^{-c} \|Z_k\|_{0,r_k} \|Z_k\|_{2s+1,r_k} \leq \\ \leq Cr^{-c} t_k^{-\alpha} \|Z_k\|_{s,r_k}^{\frac{p-(2s+1)}{p-s}} \|Z_k\|_{p,r_k}^{\frac{s+1}{p-s}} \leq Cr^{-c} t_k^{-\alpha(1+\frac{p-(3s+2)}{p-s})}.$$

Since $-\alpha(1+\frac{p-(3s+2)}{p-s}) \leq -\frac{3}{2}\alpha-1$, these two inequalities imply that

$$(30) \quad \|U_k\|_{s,r_k} \leq Cr^{-c} t_k^{-\frac{3}{2}\alpha-1}.$$

Using (25), we bound the C^0 -norm of U_k by

$$(31) \quad \|U_k\|_{0,r_k} \leq \|Z_k\|_{0,r_k} + \|[\pi, X_k]\|_{0,r_k} \leq t_k^{-\alpha} + Cr^{-c} \|X_k\|_{1,r_k} \leq Cr^{-c} t_k^{2-\alpha}.$$

By (28), (29), (30), (31) and $s+4-2\alpha = -\frac{3}{2}\alpha-1$, (a_{k+1}) follows

$$\|Z_{k+1}\|_{s,r_{k+1}} \leq Cr^{-c} (t_k^{s+4-2\alpha} + t_k^{-\frac{3}{2}\alpha-1}) \leq \\ \leq Cr^{-c} t_k^{-\frac{3}{2}\alpha-1} \leq (r^{-c}C/t_0) t_k^{-\frac{3}{2}\alpha} \leq t_{k+1}^{-\alpha}.$$

This finishes the induction.

Using (27), for every $n \geq 1$, we find $k_n \geq 0$, such that

$$\|Z_{k+1}\|_{n,r_{k+1}} \leq t_k^{s+3} \|Z_k\|_{n,r_k}, \quad \forall k \geq k_n.$$

Iterating this, we obtain

$$t_k^{s+3} \|Z_k\|_{n,r_k} \leq (t_k t_{k-1} \dots t_{k_n})^{s+3} \|Z_{k_n}\|_{n,r_{k_n}}.$$

On the other hand we have that

$$t_k t_{k-1} \dots t_{k_n} = t_{k_n}^{1+\frac{3}{2}+\dots+(\frac{3}{2})^{k-k_n}} \leq t_{k_n}^{2(\frac{3}{2})^{k+1-k_n}} = t_k^3.$$

Therefore, we obtain a bound valid for all $k > k_n$

$$\|Z_k\|_{n,r_k} \leq t_k^{2(s+3)} \|Z_{k_n}\|_{n,r_{k_n}}.$$

Consider now $m > s$ and denote by $n := 4m - 3s$. Applying the interpolation inequalities, for $k > k_n$, we obtain

$$\|Z_k\|_{m,r_k} \leq C_m r^{-cm} \|Z_k\|_{s,r_k}^{\frac{n-m}{n-s}} \|Z_k\|_{n,r_k}^{\frac{m-s}{n-s}} = C_m r^{-cm} \|Z_k\|_{s,r_k}^{\frac{3}{4}} \|Z_k\|_{n,r_k}^{\frac{1}{4}} \leq \\ \leq C_m r^{-cm} t_k^{-\alpha\frac{3}{4}+2(s+3)\frac{1}{4}} \|Z_{k_n}\|_{n,r_{k_n}}^{\frac{1}{4}} = C_m r^{-cm} t_k^{-(s+6)} \|Z_{k_n}\|_{n,r_{k_n}}^{\frac{1}{4}}.$$

Using also inequality (24), for $l = s$, we obtain

$$\|X_k\|_{m,r_k} \leq C_m r^{-c_m} t_k^{s+1} \|Z_k\|_{m,r_k} \leq t_k^{-5} \left(C_m r^{-c_m} \|Z_{k_n}\|_{\frac{1}{n},r_{k_n}} \right).$$

This shows that the series $\sum_{k \geq 0} \|X_k\|_{m,r_k}$ converges for all m . By Lemma 3.10, also $\sum_{k \geq 0} d(\varphi_{X_k})_{m,r_{k+1}}$ converges for all m and, moreover, by (26), we have that

$$\sigma_1 := \sum_{k \geq 1} d(\varphi_{X_k})_{1,r_{k+1}} \leq C r^{-c} \sum_{k \geq 1} \|X_k\|_{1,r_k} \leq C r^{-c} t_0^{-4} \sum_{k \geq 1} \epsilon_k \leq t_0^{-3}.$$

So, we may assume that $\sigma_1 \leq \theta$ and $d(\varphi_{X_k})_{1,r_{k+1}} < 1$. Then, by applying Lemma 3.8 we conclude that the sequence $\psi_{k|r}$ converges uniformly in all C^m -norms to a map $\psi : E_r \rightarrow E_R$ in \mathcal{U}_r which satisfies

$$d(\psi)_{1,r} \leq e^{C r^{-c} \sigma_1} C r^{-c} \sigma_1 \leq e^{t_0^{-2}} t_0^{-2} \leq C t_0^{-2} \leq t_0^{-1}.$$

So (22) holds, and we can also assume that $d(\psi)_{1,r} < \theta$, which, by Lemma 3.6, implies that ψ is a local diffeomorphism. Since $\psi_{k|r}$ converges in the C^1 -topology to ψ and $\psi_k^*(\tilde{\pi}) = (d\psi_k)^{-1}(\tilde{\pi}_{\psi_k})$, it follows that $\psi_k^*(\tilde{\pi})|_r$ converges in the C^0 -topology to $\psi^*(\tilde{\pi})$. On the other hand, $Z_{k|r} = \psi_k^*(\tilde{\pi})|_r - \pi|_r$ converges to 0 in the C^0 -norm, hence $\psi^*(\tilde{\pi}) = \pi|_r$. So ψ is a Poisson map and a local diffeomorphism between

$$\psi : (E_r, \pi|_r) \longrightarrow (E_R, \tilde{\pi}).$$

For **Procedure P₁**, as noted before the proposition, the first jet of φ_{X_k} is that of the identity along S . This clearly holds also for ψ_k , and since $\psi_{k|r}$ converges to ψ in the C^1 -topology, we have that ψ is also the identity along S up to first order. \square

We are ready now to finish the proof of Theorem 2.

3.5. Proof of part (a) of Theorem 2. We have to check the properties from the definition of C^p - C^1 -rigidity. Consider $U := \text{int}(E_\rho)$, for some $\rho \in (0, 1)$ and let $O \subset U$ be an open such that $S \subset O \subset \overline{O} \subset U$. Let $r < R$ be such that $O \subset E_r \subset E_R \subset U$. With d and δ from Proposition 3.12, we let

$$\mathcal{V}_O := \{W \in \mathfrak{X}^2(U) : \|W|_R - \pi|_R\|_{p,R} < \delta(r(R-r))^d\}.$$

For $\tilde{\pi} \in \mathcal{V}_O$, define $\psi_{\tilde{\pi}}$ to be the restriction to \overline{O} of the map ψ , obtained by applying **Procedure P₀** to $\tilde{\pi}|_R$. Then ψ is a Poisson diffeomorphism $(O, \pi|_O) \rightarrow (U, \tilde{\pi})$, and by (22), the assignment $\tilde{\pi} \mapsto \psi$ has the required continuity property.

3.6. Proof of part (b) of Theorem 2. Consider $\tilde{\pi}$ a Poisson structure on some neighborhood of S with $j_{|S}^1 \tilde{\pi} = j_{|S}^1 \pi$. First we show that π and $\tilde{\pi}$ are formally Poisson diffeomorphic around S . By [12], this property is controlled by the groups $H^2(A_S, \mathcal{S}^k(\nu_S^*))$. The Lie groupoid $\mathcal{G}_{|S} \rightrightarrows S$ integrates A_S and is s -connected. Since $\nu_S^* \subset A_S$ is an ideal, by Lemma B.1 from the appendix, the action of A_S on ν_S^* (hence also on $\mathcal{S}^k(\nu_S^*)$) also integrates to $\mathcal{G}_{|S}$. Since $\mathcal{G}_{|S}$ has compact s -fibers with vanishing H^2 , the Tame Vanishing Lemma implies that $H^2(A_S, \mathcal{S}^k(\nu_S^*)) = 0$. So we can apply Theorem 1.1 [12] to conclude that there exists a diffeomorphism φ between open neighborhoods of S , which is the identity on S up to first order, and such that $j_{|S}^\infty \varphi^*(\tilde{\pi}) = j_{|S}^\infty \pi$.

Let $R \in (0, 1)$ be such that $\varphi^*(\tilde{\pi})$ is defined on E_R . Using the Taylor expansion up to order $2d+1$ around S for the bivector $\pi - \varphi^*(\tilde{\pi})$ and its partial derivatives up to order p , we find a constant $M > 0$ such that

$$\|\varphi^*(\tilde{\pi})|_r - \pi|_r\|_{p,r} \leq M r^{2d+1}, \quad \forall 0 < r < R.$$

If we take $r < 2^{-d}\delta/M$, we obtain that $\|\varphi^*(\tilde{\pi})|_r - \pi|_r\|_{p,r} < \delta(r(R-r/2))^d$. So, we can apply Proposition 3.12, and **Procedure P₁** produces a Poisson diffeomorphism

$$\tau : (E_{r/2}, \pi|_{r/2}) \longrightarrow (E_r, \varphi^*(\tilde{\pi})|_r),$$

which is the identity up to first order along S . We obtain (b) with $\psi = \varphi \circ \tau$.

Remark 2. As mentioned already in the Introduction, Conn's proof has been formalized in [15, 17] into an abstract Nash Moser normal form theorem, and it is likely that one could use Theorem 6.8 [15] to prove partially our rigidity result. Nevertheless, the continuity assertion, which is important in applications (see [13]), is not a consequence of this result. There are also several technical reasons why we cannot apply [15]: we need the size of the C^p -open to depend polynomially on r^{-1} and $(R - r)^{-1}$, because we use a formal linearization argument (this dependence is not given in *loc.cit.*); to obtain diffeomorphisms which fix S , we work with vector fields which vanish along S up to first order, and it is unlikely that this Fréchet space admits smoothing operators of degree 0 (in *loc.cit.* this is the overall assumption); for the inequalities in Lemma 3.7 we need special norms for the embeddings (indexed also by “ δ ”), which are not considered in *loc.cit.*

APPENDIX A. INVARIANT TUBULAR NEIGHBORHOODS

In the proof of Theorem 2, we have used the following result:

Lemma A.1. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid with connected s -fibers and let $S \subset M$ be a compact invariant submanifold. There exists a tubular neighborhood $E \subset M$ (where $E \cong T_S M / TS$) and a metric on E , such that for all $r > 0$ the closed tube $E_r := \{v \in E : |v| \leq r\}$ is \mathcal{G} -invariant.*

The proof is inspired by the proof of Theorem 4.2 from [19], and it uses the following result (see Definition 3.11, Proposition 3.13 and Proposition 6.4 in *loc.cit.*).

Lemma A.2. *On the base of a proper Lie groupoid there exist Riemannian metrics such that every geodesic which emanates orthogonally from an orbit stays orthogonal to any orbit it passes through. Such metrics are called **adapted**.*

Proof of Lemma A.1. Let g be an adapted metric on M and let $E := TS^\perp \subset T_S M$ be the normal bundle. By rescaling g , we may assume that

- (1) the exponential is defined on E_2 and on $\text{int}(E_2)$ it is an open embedding;
- (2) for all $r \in (0, 1]$ we have that

$$\exp(E_r) = \{p \in M : d(p, S) \leq r\},$$

where d denotes the distance induced by the Riemannian structure.

Let $v \in E_1$ with $|v| := r$ and base point x . We claim that the geodesic $\gamma(t) := \exp(tv)$ at $t = 1$ is normal to $\exp(\partial E_r)$ at $\gamma(1)$:

$$T_{\gamma(1)} \exp(\partial E_r) = \dot{\gamma}(1)^\perp.$$

Let $S_r(x)$ be the sphere of radius r centered at x . By the Gauss Lemma

$$\dot{\gamma}(1)^\perp = T_{\gamma(1)} S_r(x),$$

and by (2), $\overline{B}_r(x) \subset \exp(E_r)$, where $\overline{B}_r(x)$ is the closed ball of radius r around x . Since $\overline{B}_r(x)$ and $\exp(E_r)$ intersect at $\gamma(1)$, their boundaries must be tangent at this point, and this proves the claim.

By assumption, S is a union of orbits, therefore the geodesics $\gamma(t) := \exp(tv)$, for $v \in E$, start normal to the orbits of \mathcal{G} , thus, by the property of the metric, they continue to be orthogonal to the orbits. Hence, by our claim, the orbits which intersect $\exp(\partial E_r)$ are tangent to $\exp(\partial E_r)$. By connectivity of the orbits, $\exp(\partial E_r)$ is invariant, for all $r \in (0, 1)$. Define the embedding $E \hookrightarrow M$ by

$$v \mapsto \exp(\lambda(|v|)/|v|v),$$

where $\lambda : [0, \infty) \rightarrow [0, 1)$ is a diffeomorphism which is the identity on $[0, 1/2)$. \square

APPENDIX B. INTEGRATING IDEALS

We prove that representations of a Lie algebroid A on ideals can be integrated to representations of any s -connected Lie groupoid integrating A . This result was used in the proof of part (b) of Theorem 2.

Representations of a Lie groupoid \mathcal{G} can be differentiated to representations of its Lie algebroid A , but in general, a representation of A does only integrate to a representation of the Weinstein groupoid of A (the s -fiber 1-connected) and not necessarily to one of \mathcal{G} .

We call a subbundle $I \subset A$ an **ideal**, if $\rho(I) = 0$ and $\Gamma(I)$ is an ideal of the Lie algebra $\Gamma(A)$. Using the Leibniz rule, one easily sees that, if $I \neq A$, then the second condition implies the first. An ideal I is naturally a representation of A , with A -connection given by the Lie bracket

$$\nabla_X(Y) := [X, Y], \quad X \in \Gamma(A), \quad Y \in \Gamma(I).$$

Lemma B.1. *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid with Lie algebroid A and let $I \subset A$ be an ideal. If the s -fibers of \mathcal{G} are connected, then the representation of A on I given by the Lie bracket integrates to \mathcal{G} .*

Proof. First observe that \mathcal{G} acts on the possibly singular bundle of isotropy Lie algebras $\ker(\rho) \rightarrow M$ via the formula:

$$(32) \quad g \cdot Y = \frac{d}{d\epsilon} (g \exp(\epsilon Y) g^{-1})|_{\epsilon=0}, \quad \forall Y \in \ker(\rho)_{s(g)}.$$

Let $N(I) \subset \mathcal{G}$ be the subgroupoid consisting of elements g which satisfy $g \cdot I_{s(g)} \subset I_{t(g)}$. We will prove that $N(I) = \mathcal{G}$ and that the induced action of \mathcal{G} on I differentiates to the Lie bracket.

Recall that a derivation on a vector bundle $E \rightarrow M$ (see section 3.4 in [11]) is a pair (D, V) , with D a linear operator on $\Gamma(E)$ and V a vector field on M , satisfying

$$D(f\alpha) = fD(\alpha) + V(f)\alpha, \quad \forall \alpha \in \Gamma(E), f \in C^\infty(M).$$

The flow of a derivation (D, V) , denoted by ϕ_D^t , is a vector bundle map covering the flow φ_V^t of V , $\phi_D^t(x) : E_x \rightarrow E_{\varphi_V^t(x)}$, (whenever $\varphi_V^t(x)$ is defined), which is the solution to the following differential equation

$$\phi_D^0 = \text{Id}_E, \quad \frac{d}{dt}(\phi_D^t)^*(\alpha) = (\phi_D^t)^*(D\alpha),$$

where $(\phi_D^t)^*(\alpha)_x = \phi_D^{-t}(\varphi_V^t(x))\alpha_{\varphi_V^t(x)}$.

For $X \in \Gamma(A)$, denote by $\phi^t(X, g)$ the flow of the corresponding right invariant vector field on \mathcal{G} , and by $\varphi^t(X, x)$ the flow of $\rho(X)$ on M . Conjugation by $\phi^t(X)$ is an automorphism of \mathcal{G} covering $\varphi^t(X, x)$, which we denote by

$$C(\phi^t(X)) : \mathcal{G} \longrightarrow \mathcal{G}, \quad g \mapsto \phi^t(X, t(g))g\phi^t(X, s(g))^{-1}.$$

Since $C(\phi^t(X))$ sends the s -fiber over x to the s -fiber over $\varphi^t(X, x)$, its differential at the identity 1_x gives an isomorphism

$$\text{Ad}(\phi^t(X)) : A_x \longrightarrow A_{\varphi^t(X, x)}, \quad \text{Ad}(\phi^t(X))_x := dC(\phi^t(X))|_{A_x}.$$

On $\ker(\rho)_x$, we recover the action (32) of $g = \phi^t(X, x)$. We have that:

$$(33) \quad \begin{aligned} \frac{d}{dt}(\text{Ad}(\phi^t(X))^*Y)_x &= \frac{d}{dt}\text{Ad}(\phi^{-t}(X, \varphi^t(X, x)))Y_{\varphi^t(X, x)} = \\ &= -\frac{d}{ds}(\text{Ad}(\phi^{-t}(X, \varphi^t(X, x)))\text{Ad}(\phi^s(X, \varphi^{t-s}(X, x)))Y_{\varphi^{t-s}(X, x)})|_{s=0} = \\ &= \text{Ad}(\phi^{-t}(X, \varphi^t(X, x)))[X, Y]_{\varphi^t(X, x)} = \text{Ad}(\phi^t(X))^*([X, Y])_x, \end{aligned}$$

for $Y \in \Gamma(A)$, where we have used the adjoint formulas from Proposition 3.7.1 in [11]. This shows that $\text{Ad}(\phi^t(X))$ is the flow of the derivation $([X, \cdot], \rho(X))$ on A .

Since I is an ideal, the derivation $[X, \cdot]$ restricts to a derivation on I , and therefore I is invariant under $Ad(\phi^t(X))$. This proves that for all $Y \in I_x$,

$$Ad(\phi^t(X, x))Y = \phi^t(X, x) \cdot Y \in I.$$

So $N(I)$ contains all the elements in \mathcal{G} of the form $\phi^t(X, x)$. The set of such elements contains an open neighborhood O of the unit section in \mathcal{G} . Since the s -fibers of \mathcal{G} are connected, O generates \mathcal{G} (see Proposition 1.5.8 in [11]), therefore $N(I) = \mathcal{G}$ and so (32) defines an action of \mathcal{G} on I .

Using that $\phi^{-t}(X, \phi^t(X, x)) = \phi^t(X, x)^{-1}$, equation (33) gives

$$\frac{d}{dt}\bigg|_{t=0} (\phi^t(X, x)^{-1} \cdot Y_{\phi^t(X, x)}) = [X, Y]_x, \quad \forall X \in \Gamma(A), Y \in \Gamma(I).$$

Thus, the action differentiates to the Lie bracket (see Definition 3.6.8 [11]). \square

APPENDIX C. THE TAME VANISHING LEMMA

We prove now the Tame Vanishing Lemma, an existence result for tame homotopy operators on the complex computing Lie algebroid cohomology with coefficients. We have used this lemma for the Poisson complex in the proof of Theorem 2. The Tame Vanishing Lemma is very useful when applying the Nash-Moser techniques to various geometric problems, for another example see [14].

C.1. The weak C^∞ -topology. The compact-open C^k -topology on the space of sections of a vector bundle can be generated by a family of semi-norms, and we recall here a construction of such semi-norms, generalizing the construction from section 3. These semi-norms will be used to express the tameness property of the homotopy operators.

Let $W \rightarrow M$ be a vector bundle. Consider $\mathcal{U} := \{U_i\}_{i \in I}$ a locally finite open cover of M by relatively compact domains of coordinate charts $\{\chi_i : U_i \xrightarrow{\sim} \mathbb{R}^m\}_{i \in I}$ and choose trivializations for $W|_{U_i}$. Let $\mathcal{O} := \{O_i\}_{i \in I}$ be a second open cover, with $\overline{O_i}$ compact and $\overline{O_i} \subset U_i$. A section $\sigma \in \Gamma(W)$ can be represented in these charts by a family of smooth functions $\{\sigma_i : \mathbb{R}^m \rightarrow \mathbb{R}^k\}_{i \in I}$, where k is the rank of W . For $U \subset M$, an open set with compact closure, we have that \overline{U} intersects only a finite number of the coordinate charts U_i . Denote the set of such indexes by $I_U \subset I$. Define the n -th norm of σ on U by

$$\|\sigma\|_{n, \overline{U}} := \sup\{|\frac{\partial^{|\alpha|}}{\partial x^\alpha} \sigma_i(x)| : |\alpha| \leq n, x \in \chi_i(U \cap O_i), i \in I_U\}.$$

For a fixed n , the family of semi-norms $\|\cdot\|_{n, \overline{U}}$, with U a relatively compact open in M , generate the **compact-open C^n -topology** on $\Gamma(W)$. The union of all these topologies, for $n \geq 0$, is called the **weak C^∞ -topology** on $\Gamma(W)$. Observe that the semi-norms $\{\|\cdot\|_{n, \overline{U}}\}_{n \geq 0}$ induce norms on $\Gamma(W|_{\overline{U}})$.

C.2. The statement of the Tame Vanishing Lemma.

Lemma C.1 (The Tame Vanishing Lemma). *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid with Lie algebroid A and let V be a representation of \mathcal{G} . If the s -fibers of \mathcal{G} are compact and their de Rham cohomology vanishes in degree p , then*

$$H^p(A, V) = 0.$$

Moreover, there exist linear homotopy operators

$$\Omega^{p-1}(A, V) \xleftarrow{h_1} \Omega^p(A, V) \xleftarrow{h_2} \Omega^{p+1}(A, V),$$

$$d_\nabla h_1 + h_2 d_\nabla = \text{Id},$$

which satisfy

(1) *invariant locality: for every orbit O of A , they induce linear maps*

$$\Omega^{p-1}(A|_O, V|_O) \xleftarrow{h_{1,O}} \Omega^p(A|_O, V|_O) \xleftarrow{h_{2,O}} \Omega^{p+1}(A|_O, V|_O),$$

such that for all $\omega \in \Omega^p(A, V)$, $\eta \in \Omega^{p+1}(A, V)$, we have that

$$h_{1,O}(\omega|_O) = (h_1\omega)|_O, \quad h_{2,O}(\eta|_O) = (h_2\eta)|_O,$$

(2) *tameness: for every invariant open $U \subset M$, with \overline{U} compact, there are constants $C_{n,U} > 0$, such that*

$$\|h_1(\omega)\|_{n,\overline{U}} \leq C_{n,U} \|\omega\|_{n+s,\overline{U}}, \quad \|h_2(\eta)\|_{n,\overline{U}} \leq C_{n,U} \|\eta\|_{n+s,\overline{U}},$$

for all $\omega \in \Omega^p(A|_{\overline{U}}, V|_{\overline{U}})$ and $\eta \in \Omega^{p+1}(A|_{\overline{U}}, V|_{\overline{U}})$, where

$$s = \lfloor \frac{1}{2} \text{rank}(A) \rfloor + 1.$$

We also note the following consequences of the proof:

Corollary C.2. *The constants $C_{n,U}$ can be chosen such that they are uniform over all invariant open subsets of U . More precisely: if $V \subset U$ is a second invariant open, then one can choose $C_{n,V} := C_{n,U}$, assuming that the norms on \overline{U} and \overline{V} are computed using the same charts and trivializations.*

Corollary C.3. *The homotopy operators preserve the order of vanishing around orbits. More precisely: if O is an orbit of A , and $\omega \in \Omega^p(A, V)$ is a form such that $j_O^k \omega = 0$, then $j_O^k h_1(\omega) = 0$; and similarly for h_2 .*

C.3. The de Rham complex of a fiber bundle. To prove the Tame Vanishing Lemma, we first construct tame homotopy operators for the foliated de Rham complex of a fiber bundle. For this, we use a result on the family of inverses of elliptic operators (Proposition C.7), which we prove at the end of the section.

Let $\pi : \mathcal{B} \rightarrow M$ be a locally trivial fiber bundle whose fibers $\mathcal{B}_x := \pi^{-1}(x)$ are diffeomorphic to a compact, connected manifold F and let $V \rightarrow M$ be a vector bundle. The space of vertical vectors on \mathcal{B} will be denoted by $T^\pi \mathcal{B}$ and the space of foliated forms with values in $\pi^*(V)$ by $\Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V))$. An element $\omega \in \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V))$ is a smooth family of forms on the fibers of π with values in V

$$\omega = \{\omega_x\}_{x \in M}, \quad \omega_x \in \Omega^\bullet(\mathcal{B}_x, V_x).$$

The fiberwise exterior derivative induces the differential

$$d \otimes I_V : \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)) \longrightarrow \Omega^{\bullet+1}(T^\pi \mathcal{B}, \pi^*(V)),$$

$$d \otimes I_V(\omega)_x := (d \otimes I_{V_x})(\omega_x), \quad x \in M.$$

We construct the homotopy operators using Hodge theory. Let m be a metric on $T^\pi \mathcal{B}$, or equivalently a smooth family of Riemannian metrics $\{m_x\}_{x \in M}$ on the fibers of π . Integration against the volume density gives an inner product on $\Omega^\bullet(\mathcal{B}_x)$

$$(\eta, \theta) := \int_{\mathcal{B}_x} m_x(\eta, \theta) |dVol(m_x)|, \quad \eta, \theta \in \Omega^q(\mathcal{B}_x).$$

Let δ_x denote the formal adjoint of d with respect to this inner product

$$\delta_x : \Omega^{\bullet+1}(\mathcal{B}_x) \longrightarrow \Omega^\bullet(\mathcal{B}_x),$$

i.e. δ_x is the unique linear first order differential operator satisfying

$$(d\eta, \theta) = (\eta, \delta_x \theta), \quad \forall \eta \in \Omega^\bullet(\mathcal{B}_x), \theta \in \Omega^{\bullet+1}(\mathcal{B}_x).$$

The Laplace-Beltrami operator associated to m_x will be denoted by

$$\Delta_x : \Omega^\bullet(\mathcal{B}_x) \longrightarrow \Omega^\bullet(\mathcal{B}_x), \quad \Delta_x := d\delta_x + \delta_x d.$$

Both these operators induce linear differential operators on $\Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V))$

$$\delta \otimes I_V : \Omega^{\bullet+1}(T^\pi \mathcal{B}, \pi^*(V)) \rightarrow \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)), \quad \delta \otimes I_V(\omega)_x := (\delta_x \otimes I_{V_x})(\omega_x),$$

$$\Delta \otimes I_V : \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)) \rightarrow \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)), \quad \Delta \otimes I_V(\omega)_x := (\Delta_x \otimes I_{V_x})(\omega_x).$$

By the Hodge theorem, if the fiber F of \mathcal{B} has vanishing de Rham cohomology in degree p , then Δ_x is invertible in degree p .

Lemma C.4. *If $H^p(F) = 0$ then the following hold:*

(a) $\Delta \otimes I_V$ is invertible in degree p and its inverse is given by

$$G \otimes I_V : \Omega^p(T^\pi \mathcal{B}, \pi^*(V)) \longrightarrow \Omega^p(T^\pi \mathcal{B}, \pi^*(V)), \\ (G \otimes I_V)(\omega)_x := (\Delta_x^{-1} \otimes I_{V_x})(\omega_x), \quad x \in M;$$

(b) the maps $H_1 := (\delta \otimes I_V) \circ (G \otimes I_V)$ and $H_2 := (G \otimes I_V) \circ (\delta \otimes I_V)$

$$\Omega^{p-1}(T^\pi \mathcal{B}, \pi^*(V)) \xleftarrow{H_1} \Omega^p(T^\pi \mathcal{B}, \pi^*(V)) \xleftarrow{H_2} \Omega^{p+1}(T^\pi \mathcal{B}, \pi^*(V))$$

are linear homotopy operators in degree p ;

(c) H_1 and H_2 satisfy the following local-tameness property: for every relatively compact open $U \subset M$, there are constants $C_{n,U} > 0$ such that

$$\|H_1(\eta)\|_{n, \mathcal{B}_{|\overline{U}}} \leq C_{n,U} \|\eta\|_{n+s, \mathcal{B}_{|\overline{U}}}, \quad \forall \eta \in \Omega^p(T^\pi \mathcal{B}_{|\overline{U}}, \pi^*(V_{|\overline{U}})),$$

$$\|H_2(\omega)\|_{n, \mathcal{B}_{|\overline{U}}} \leq C_{n,U} \|\omega\|_{n+s, \mathcal{B}_{|\overline{U}}}, \quad \forall \omega \in \Omega^{p+1}(T^\pi \mathcal{B}_{|\overline{U}}, \pi^*(V_{|\overline{U}})).$$

where $s = \lfloor \frac{1}{2} \dim(F) \rfloor + 1$. Moreover, if $V \subset U$, then one can take $C_{n,V} := C_{n,U}$.

Proof. In a trivialization chart the operator $\Delta \otimes I_V$ is given by a smooth family of Laplace-Beltrami operators:

$$\Delta_x : \Omega^p(F)^k \longrightarrow \Omega^p(F)^k,$$

where k is the rank of V . These operators are elliptic and invertible, therefore, by Proposition C.7, $\Delta_x^{-1}(\omega_x)$ is smooth in x , for every smooth family $\omega_x \in \Omega^p(F)^k$. This shows that $G \otimes I_V$ maps smooth forms to smooth forms. Clearly $G \otimes I_V$ is the inverse of $\Delta \otimes I_V$, so we have proven (a).

For part (c), let $U \subset M$ be a relatively compact open. Applying part (2) of Proposition C.7 to a family of coordinate charts which cover \overline{U} , we find constants $D_{n,U}$ such that

$$\|G \otimes I_V(\eta)\|_{n, \mathcal{B}_{|\overline{U}}} \leq D_{n,U} \|\eta\|_{n+s-1, \mathcal{B}_{|\overline{U}}}, \quad \forall \eta \in \Omega^p(T^\pi \mathcal{B}_{|\overline{U}}, \pi^*(V_{|\overline{U}})).$$

Moreover, the constants can be chosen such that they are decreasing in U . Since H_1 and H_2 are defined as the composition of $G \otimes I_V$ with a linear differential operator of degree one, it follows that we can also find constants $C_{n,U}$ such that the inequalities from (c) are satisfied, and which are also decreasing in U .

For part (b), using that $\delta_x^2 = 0$, we obtain that Δ_x commutes with $d\delta_x$

$$\Delta_x d\delta_x = (d\delta_x + \delta_x d)\delta_x = d\delta_x d\delta_x + \delta_x d^2\delta_x = d\delta_x d\delta_x,$$

$$d\delta_x \Delta_x = d\delta_x (d\delta_x + \delta_x d) = d\delta_x d\delta_x + d\delta_x^2 d = d\delta_x d\delta_x.$$

This implies that $\Delta \otimes I_V$ commutes with $(d \otimes I_V)(\delta \otimes I_V)$, and thus $G \otimes I_V$ commutes with $(d \otimes I_V)(\delta \otimes I_V)$. Using this, we obtain that H_1 and H_2 are homotopy operators

$$\begin{aligned} I &= (G \otimes I_V)(\Delta \otimes I_V) = (G \otimes I_V)((d \otimes I_V)(\delta \otimes I_V) + (\delta \otimes I_V)(d \otimes I_V)) = \\ &= (d \otimes I_V)(\delta \otimes I_V)(G \otimes I_V) + (G \otimes I_V)(\delta \otimes I_V)(d \otimes I_V) = \\ &= (d \otimes I_V)H_1 + H_2(d \otimes I_V). \end{aligned}$$

□

C.4. Proof of the Tame Vanishing Lemma. Let $\mathcal{G} \rightrightarrows M$ be as in the statement. By passing to the connected components of the identities in the s -fibers [16], we may assume that \mathcal{G} is s -connected. Then $s : \mathcal{G} \rightarrow M$ is a locally trivial fiber bundle with compact fibers whose cohomology vanishes in degree p . We will apply Lemma C.4 to the complex of s -foliated forms with coefficients in $s^*(V)$

$$(\Omega^\bullet(T^s\mathcal{G}, s^*(V)), d \otimes I_V).$$

Recall that the **right translation** by an arrow $g \in \mathcal{G}$ is the diffeomorphism between the s -fibers above $y = t(g)$ and above $x = s(g)$, given by:

$$r_g : \mathcal{G}_y \xrightarrow{\sim} \mathcal{G}_x, \quad r_g(h) := hg.$$

A form $\omega \in \Omega^\bullet(T^s\mathcal{G}, s^*(V))$ is called **invariant**, if it satisfies

$$(r_g^* \otimes g)(\omega_{hg}) = \omega_h, \quad \forall h, g \in \mathcal{G}, \text{ with } s(h) = t(g),$$

where $r_g^* \otimes g$ is the linear isomorphism $\eta \mapsto g \cdot \eta \circ dr_g$. Denote the space of invariant V -valued forms on \mathcal{G} by $\Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$.

It is well-known that forms on A with values in V are in one to one correspondence with invariant V -valued forms on \mathcal{G} ; this correspondence is given by

$$J : \Omega^\bullet(A, V) \longrightarrow \Omega^\bullet(T^s\mathcal{G}, s^*(V)), \quad J(\eta)_g := (r_{g^{-1}}^* \otimes g^{-1})(\eta_{t(g)}).$$

The map J is also a chain map, thus it induces an isomorphism of complexes (see Theorem 1.2 [23] and also [14] for coefficients)

$$(34) \quad J : (\Omega^\bullet(A, V), d_\nabla) \xrightarrow{\sim} (\Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}, d \otimes I_V).$$

A left inverse for J (i.e. a map P such that $P \circ J = \text{Id}$) is given by

$$P : \Omega^\bullet(T^s\mathcal{G}, s^*(V)) \longrightarrow \Omega^\bullet(A, V), \quad P(\omega)_x := \omega_{u(x)}.$$

Let $\langle \cdot, \cdot \rangle$ be an inner product on A . Using right translations, we extend $\langle \cdot, \cdot \rangle$ to an invariant metric m on $T^s\mathcal{G}$:

$$m(X, Y)_g := \langle dr_{g^{-1}}X, dr_{g^{-1}}Y \rangle_{t(g)}, \quad \forall X, Y \in T_g^s\mathcal{G}.$$

Invariance of m implies that the right translation by an arrow $g : x \rightarrow y$ is an isometry between the s -fibers

$$r_g : (\mathcal{G}_y, m_y) \xrightarrow{\sim} (\mathcal{G}_x, m_x).$$

The corresponding operators from subsection C.3 are also invariant.

Lemma C.5. *The operators $\delta \otimes I_V$, $\Delta \otimes I_V$, H_1 and H_2 , corresponding to m , send invariant forms to invariant forms.*

Proof. Since right translations are isometries and the operators δ_z are invariant under isometries we have that $r_g^* \circ \delta_x = \delta_y \circ r_g^*$, for all arrows $g : x \rightarrow y$.

For $\eta \in \Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$ we have that

$$\begin{aligned} (r_g^* \otimes g)(\delta \otimes I_V(\eta))|_{\mathcal{G}_x} &= (r_g^* \circ \delta_x \otimes g)(\eta|_{\mathcal{G}_x}) = (\delta_y \circ r_g^* \otimes g)(\eta|_{\mathcal{G}_x}) = \\ &= (\delta_y \otimes I_{V_y})(r_g^* \otimes g)(\eta|_{\mathcal{G}_x}) = (\delta_y \otimes I_{V_y})(\eta|_{\mathcal{G}_y}) = (\delta \otimes I_V)(\eta)|_{\mathcal{G}_y}. \end{aligned}$$

This shows that $\delta \otimes I_V(\eta) \in \Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$. The other operators are constructed in terms of $\delta \otimes I_V$ and $d \otimes I_V$, thus they also preserve $\Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$. \square

This lemma and the isomorphism (34) imply that the maps

$$\Omega^{p-1}(A, V) \xleftarrow{h_1} \Omega^p(A, V) \xleftarrow{h_2} \Omega^{p+1}(A, V),$$

$$h_1 := P \circ H_1 \circ J, \quad h_2 := P \circ H_2 \circ J,$$

are linear homotopy operators for the Lie algebroid complex in degree p .

For part (1) of the Tame Vanishing Lemma, let $\omega \in \Omega^p(A, V)$ and $O \subset M$ an orbit of A . Since \mathcal{G} is s -connected we have that $s^{-1}(O) = t^{-1}(O) = \mathcal{G}|_O$. Clearly

$J(\omega)|_{s^{-1}(O)}$ depends only on $\omega|_O$. By the construction of H_1 , for all $x \in O$, we have that

$$h_1(\omega)_x = H_1(J(\omega))_{1_x} = (\delta_x \circ \Delta_x^{-1} \otimes I_{V_x})(J(\omega)|_{s^{-1}(x)})_{1_x}.$$

Thus $h_1(\omega)|_O$ depends only on $\omega|_O$. The same argument applies also to h_2 .

Before checking part (2), we give a simple lemma:

Lemma C.6. *Consider a vector bundle map $A : F_1 \rightarrow F_2$ between vector bundles $F_1 \rightarrow M_1$ and $F_2 \rightarrow M_2$, covering a map $f : M_1 \rightarrow M_2$. If A is fiberwise invertible and f is proper, then the pullback map*

$$A^* : \Gamma(F_2) \longrightarrow \Gamma(F_1), \quad A(\sigma)_x := A_x^{-1}(\sigma_{f(x)})$$

satisfies the following tameness inequalities: for every open $U \subset M_2$, with \overline{U} compact, there are constants $C_{n,U} > 0$ such that

$$\|A^*(\sigma)\|_{n, \overline{f^{-1}(U)}} \leq C_{n,U} \|\sigma\|_{n,U}, \quad \forall \sigma \in \Gamma(F_2|_{\overline{U}}).$$

Moreover:

- (a) *if $U' \subset U$ is open, and one uses the same charts when computing the norms, then one can choose $C_{n,U'} := C_{n,U}$;*
- (b) *if $N \subset M_2$ is a submanifold and $\sigma \in \Gamma(F_2)$ satisfies $j_N^k(\sigma) = 0$, then its pullback satisfies $j_{f^{-1}(N)}^k(A^*(\sigma)) = 0$.*

Proof. Since A is fiberwise invertible, we can assume that $F_1 = f^*(F_2)$ and $A^* = f^*$. By choosing a vector bundle F' such that $F_2 \oplus F'$ is trivial, we reduce the problem to the case when F_2 is the trivial line bundle. So, we have to check that $f^* : C^\infty(M_2) \rightarrow C^\infty(M_1)$ has the desired properties. But this is straightforward: we just cover both $\overline{f^{-1}(U)}$ and \overline{U} by charts, and apply the chain rule. The constants $C_{n,U}$ are the C^n -norm of f over $\overline{f^{-1}(U)}$, and therefore are getting smaller if U gets smaller. This implies (a). For part (b), just observe that $j_{f(x)}^k(\sigma) = 0$ implies $j_x^k(\sigma \circ f) = 0$. \square

Part (2) of the Tame Vanishing Lemma follows by Lemma C.4 (c) and by applying Lemma C.6 to J and P . Corollary C.2 follows from Lemma C.6 (a) and Lemma C.4 (c). To prove Corollary C.3, consider ω a form with $j_O^k \omega = 0$, for O an orbit. Then, by Lemma C.6 (b), it follows that $J(\omega)$ vanishes up to order k along $t^{-1}(O) = \mathcal{G}_O$. By construction, we have that H_1 is $C^\infty(M)$ linear, therefore also $H_1(J(\omega))$ vanishes up to order k along \mathcal{G}_O ; and again by Lemma C.6 (b) $h_1(\omega) = u^*(H_1(J(\omega)))$ vanishes along $O = u^{-1}(\mathcal{G}_O)$ up to order k .

C.5. The inverse of a family of elliptic operators. This subsection is devoted to proving the following result:

Proposition C.7. *Consider a smooth family of linear differential operators*

$$P_x : \Gamma(V) \longrightarrow \Gamma(W), \quad x \in \mathbb{R}^m,$$

between sections of vector bundles V and W over a compact base F . If P_x is elliptic of degree $d \geq 1$ and invertible for all $x \in \mathbb{R}^m$, then

- (1) *the family of inverses $\{Q_x := P_x^{-1}\}_{x \in \mathbb{R}^m}$ induces a linear operator*

$$Q : \Gamma(p^*(W)) \longrightarrow \Gamma(p^*(V)), \quad \{\omega_x\}_{x \in \mathbb{R}^m} \mapsto \{Q_x \omega_x\}_{x \in \mathbb{R}^m},$$

where $p^(V) := V \times \mathbb{R}^m \rightarrow F \times \mathbb{R}^m$ and $p^*(W) := W \times \mathbb{R}^m \rightarrow F \times \mathbb{R}^m$;*

- (2) *Q is locally tame, in the sense that for all bounded opens $U \subset \mathbb{R}^m$, there exist constants $C_{n,U} > 0$, such that the following inequalities hold*

$$\|Q(\omega)\|_{n, F \times \overline{U}} \leq C_{n,U} \|\omega\|_{n+s-1, F \times \overline{U}}, \quad \forall \omega \in \Gamma(p^*(W)|_{F \times \overline{U}}),$$

with $s = \lfloor \frac{1}{2} \dim(F) \rfloor + 1$. If $U' \subset U$, then one can take $C_{n,U'} := C_{n,U}$.

Fixing C^n -norms $\|\cdot\|_n$ on $\Gamma(V)$, we induce semi-norms on $\Gamma(p^*(V))$:

$$\|\omega\|_{n, F \times \overline{U}} := \sup_{0 \leq k+|\alpha| \leq n} \sup_{x \in U} \left\| \frac{\partial^{|\alpha|} \omega_x}{\partial x^\alpha} \right\|_k,$$

where $\omega \in \Gamma(p^*(V))$ is regarded as a smooth family $\omega = \{\omega_x \in \Gamma(V)\}_{x \in \mathbb{R}^m}$. Similarly, fixing norms on $\Gamma(W)$, we define also norms on $\Gamma(p^*(W))$.

Endow $\Gamma(V)$ and $\Gamma(W)$ also with Sobolev norms, denoted by $\{|\cdot|_n\}_{n \geq 0}$. Loosely speaking, $|\omega|_n$ measures the L^2 -norm of ω and its partial derivatives up to order n (for a precise definition see e.g. [9]). Denote by $H_n(\Gamma(V))$ and by $H_n(\Gamma(W))$ the completion of $\Gamma(V)$, respectively of $\Gamma(W)$, with respect to the Sobolev norm $|\cdot|_n$.

We will use the standard inequalities between the Sobolev and the C^n -norms, which follow from the Sobolev embedding theorem

$$(35) \quad \|\omega\|_n \leq C_n |\omega|_{n+s}, \quad |\omega|_n \leq C_n \|\omega\|_n,$$

for all $\omega \in \Gamma(V)$ (resp. $\Gamma(W)$), where $s = \lfloor \frac{1}{2} \dim(F) \rfloor + 1$ and $C_n > 0$ are constants.

Since P_x is of order d , it induces continuous linear maps between the Sobolev spaces, denoted by

$$[P_x]_n : H_{n+d}(\Gamma(V)) \longrightarrow H_n(\Gamma(W)).$$

These maps are invertible.

Lemma C.8. *If an elliptic differential operator of degree d*

$$P : \Gamma(V) \longrightarrow \Gamma(W)$$

is invertible, then for every $n \geq 0$ the induced map

$$[P]_n : H_{n+d}(\Gamma(V)) \longrightarrow H_n(\Gamma(W))$$

is also invertible and its inverse is induced by the inverse of P .

Proof. Since P is elliptic, it is invertible modulo smoothing operators (see Lemma 1.3.5 in [9]), i.e. there exists a pseudo-differential operator

$$\Psi : \Gamma(W) \longrightarrow \Gamma(V),$$

of degree $-d$ such that $\Psi P - \text{Id} = K_1$ and $P\Psi - \text{Id} = K_2$, where K_1 and K_2 are smoothing operators. Since Ψ is of degree $-d$, it induces continuous maps

$$[\Psi]_n : H_n(\Gamma(W)) \longrightarrow H_{n+d}(\Gamma(V)),$$

and since K_1 and K_2 are smoothing operators, they induce continuous maps

$$[K_1]_n : H_n(\Gamma(V)) \longrightarrow \Gamma(V), \quad [K_2]_n : H_n(\Gamma(W)) \longrightarrow \Gamma(W).$$

We show now that $[P]_n$ is a bijection:

injective: For $\eta \in H_{n+d}(\Gamma(V))$, with $[P]_n \eta = 0$, we have that

$$\eta = (\text{Id} - [\Psi]_n [P]_n) \eta = -[K_1]_n \eta \in \Gamma(V),$$

hence $[P]_n \eta = P\eta$. By injectivity of P , we have that $\eta = 0$.

surjective: For $\theta \in H_n(\Gamma(W))$, we have that

$$([P]_n [\Psi]_n - \text{Id}) \theta = [K_2]_n \theta \in \Gamma(W),$$

and, since P is onto, $[K_2]_n \theta = P\eta$ for some $\eta \in \Gamma(V)$. So θ is in the range of $[P]_n$:

$$\theta = [P]_n ([\Psi]_n \theta - \eta).$$

The inverse of a bounded operator between Banach spaces is bounded, therefore

$$|[P]_n^{-1} \theta|_{n+d} \leq C_n |\theta|_n, \quad \theta \in H_n(\Gamma(W)),$$

for some $C_n > 0$. For $\theta \in \Gamma(W)$, this show that P^{-1} induces continuous maps

$$[P^{-1}]_n : H_n(\Gamma(W)) \longrightarrow H_{n+d}(\Gamma(V)).$$

Now $[P^{-1}]_n$ and $[P]_n^{-1}$ coincide on (the dense) $\Gamma(W)$, so they must be equal. \square

For two Banach spaces B_1 and B_2 denote by $Lin(B_1, B_2)$ the Banach space of bounded linear maps between them and by $Iso(B_1, B_2)$ the open subset consisting of invertible maps. The following proves that the family $[P_x]_n$ is smooth.

Lemma C.9. *Let $\{P_x\}_{x \in \mathbb{R}^m}$ be a smooth family of linear differential operators of order d between the sections of vector bundles V and W , both over a compact manifold F . Then the map induced by P from \mathbb{R}^m to the space of bounded linear operators between the Sobolev spaces*

$$\mathbb{R}^m \ni x \mapsto [P_x]_n \in Lin(H_{n+d}(\Gamma(V)), H_n(\Gamma(W)))$$

is smooth and its derivatives are induced by the derivatives of P_x .

Proof. Linear differential operators of degree d from V to W are sections of the vector bundle $Hom(J^d(V); W) = J^d(V)^* \otimes W$, where $J^d(V) \rightarrow F$ is the d -th jet bundle of V . Therefore, P can be viewed as a smooth section of the pullback bundle $p^*(Hom(J^d(V); W)) := Hom(J^d(V); W) \times \mathbb{R}^m \rightarrow F \times \mathbb{R}^m$. Since F is compact, by choosing a partition of unity on F with supports inside some opens on which V and W trivialize, one can write any section of $p^*(Hom(J^d(V); W))$ as a linear combination of sections of $Hom(J^d(V); W)$ with coefficients in $C^\infty(\mathbb{R}^m \times F)$. Hence, there are constant differential operators P_i and $f_i \in C^\infty(\mathbb{R}^m \times F)$, such that

$$P_x = \sum f_i(x) P_i.$$

So it suffices to prove that for $f \in C^\infty(\mathbb{R}^m \times F)$, multiplication with $f(x)$

$$\mu(f(x)) : \Gamma(W) \longrightarrow \Gamma(W),$$

induces a smooth map between

$$\mathbb{R}^m \ni x \mapsto [\mu(f(x))]_n \in Lin(H_n(\Gamma(W)), H_n(\Gamma(W))).$$

First, it is easy to see that for any smooth function $g \in C^\infty(\mathbb{R}^m \times F)$ and every compact $K \subset \mathbb{R}^m$, there are constants $C_n(g, K)$ such that $|g(x)\sigma|_n \leq C_n(g, K)|\sigma|_n$ for all $x \in K$ and $\sigma \in H_n(\Gamma(W))$; or equivalently that the operator norm satisfies $||[\mu(g(x))]_n|_{op} \leq C_n(g, K)$, for $x \in K$. Consider now $f \in C^\infty(\mathbb{R}^m \times F)$ and $\bar{x} \in \mathbb{R}^m$. Using the Taylor expansion of f at \bar{x} , write

$$f(x) - f(\bar{x}) - \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\bar{x})(x_i - \bar{x}_i) = \sum_{1 \leq i \leq j \leq m} (x_i - \bar{x}_i)(x_j - \bar{x}_j) T_{\bar{x}}^{i,j}(x),$$

where $T_{\bar{x}}^{i,j} \in C^\infty(\mathbb{R}^m \times F)$. This implies that

$$\begin{aligned} |[\mu(f(x))]_n - [\mu(f(\bar{x}))]_n - \sum_{i=1}^m [\mu(\frac{\partial f}{\partial x_i}(\bar{x}))]_n (x_i - \bar{x}_i)|_{op} &\leq \\ &\leq |x - \bar{x}|^2 \sum_{1 \leq i \leq j \leq m} C_n(T_{\bar{x}}^{i,j}, K), \end{aligned}$$

for all $x \in K$, where K is a closed ball centered at \bar{x} . This inequality shows that $[\mu(f(x))]_n$ is C^1 in x , with partial derivatives given by

$$\frac{\partial}{\partial x_i} [\mu(f)]_n = [\mu(\frac{\partial f}{\partial x_i})]_n.$$

The statement follows now inductively. \square

Proof of Proposition C.7. By Lemma C.8, $Q_x = P_x^{-1}$ induces continuous operators

$$[Q_x]_n : H_n(\Gamma(W)) \longrightarrow H_{n+d}(\Gamma(V)).$$

We claim that the following map is smooth

$$\mathbb{R}^m \ni x \mapsto [Q_x]_n \in \text{Lin}(H_n(\Gamma(W)), H_{n+d}(\Gamma(V))).$$

This follows by Lemma C.8 and Lemma C.9, since we can write

$$[Q_x]_n = [P_x^{-1}]_n = [P_x]_n^{-1} = \iota([P_x]_n),$$

where ι is the (smooth) inversion map

$$\iota : \text{Iso}(H_{n+d}(\Gamma(V)), H_n(\Gamma(W))) \longrightarrow \text{Iso}(H_n(\Gamma(W)), H_{n+d}(\Gamma(V))).$$

Let $\omega = \{\omega_x\}_{x \in \mathbb{R}^m} \in \Gamma(p^*(W))$. By our claim and Lemma C.9, it follows that

$$x \mapsto [Q_x]_n[\omega_x]_n = [Q_x\omega_x]_{n+d} \in H_{n+d}(\Gamma(V))$$

is a smooth map. On the other hand, the Sobolev inequalities (35) show that the inclusion $\Gamma(V) \rightarrow \Gamma^n(V)$, where $\Gamma^n(V)$ is the space of sections of V of class C^n (endowed with the norm $\|\cdot\|_n$), extends to a continuous map

$$H_{n+s}(\Gamma(V)) \longrightarrow \Gamma^n(V).$$

Since also evaluation $ev_p : \Gamma^n(V) \rightarrow V_p$ at $p \in F$ is continuous, it follows that the map $x \mapsto Q_x\omega_x(p) \in V_p$ is smooth. This is enough to conclude smoothness of the family $\{Q_x\omega_x\}_{x \in \mathbb{R}^m}$, so $Q(\omega) \in \Gamma(p^*(V))$. This finishes the proof of the first part.

For the second part, let $U \subset \mathbb{R}^m$ be an open with \overline{U} compact. Since the map $x \mapsto [Q_x]_n$ is smooth, it follows that the numbers are finite:

$$(36) \quad D_{n,m,U} := \sup_{x \in U} \sup_{|\alpha| \leq m} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} [Q_x]_n \right|_{op},$$

where $|\cdot|_{op}$ denotes the operator norm. Let $\omega = \{\omega_x\}_{x \in \overline{U}}$ be an element of $\Gamma(p^*(W))_{|F \times \overline{U}}$. By Lemma C.9, also the map $x \mapsto [\omega_x]_n \in H_n(\Gamma(W))$ is smooth and that for all multi-indices γ

$$\frac{\partial^{|\gamma|}}{\partial x^\gamma} [\omega_x]_n = \left[\frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right]_n.$$

Let k and α be such that $|\alpha| + k \leq n$. Using (35), (36) we obtain

$$\begin{aligned} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (Q_x\omega_x) \right\|_k &\leq \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (Q_x\omega_x) \right\|_{k+d-1} \leq C_{k+d-1} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (Q_x\omega_x) \right|_{k+s+d-1} \leq \\ &\leq C_{k+d-1} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta \gamma} \left| \frac{\partial^{|\beta|}}{\partial x^\beta} Q_x \frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right|_{k+s+d-1} \leq \\ &\leq C_{k+d-1} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta \gamma} D_{k+s-1,|\beta|,U} \left| \frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right|_{k+s-1} \leq \\ &\leq C_{k+d-1} C_{k+s-1} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta \gamma} D_{k+s-1,|\beta|,U} \left\| \frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right\|_{k+s-1} \leq \\ &\leq C_{n,U} \|\omega\|_{n+s-1,F \times \overline{U}}. \end{aligned}$$

This proves the second part:

$$\|Q(\omega)\|_{n,F \times \overline{U}} \leq C_{n,U} \|\omega\|_{n+s-1,F \times \overline{U}}.$$

The constants $D_{n,m,U}$ are clearly decreasing in U , hence for $U' \subset U$ we also have that $C_{n,U'} \leq C_{n,U}$. This finishes the proof of Proposition C.7.

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